

## SETS AND RELATIONS

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- Sets**
- Operations: intersection, union, difference, Cartesian product
  - Big  $\cup$ , powerset ( $2^A$ ), partition ( $\pi \subseteq 2^A, \emptyset \notin \pi, \forall i \neq j : \pi_i \cap \pi_j = \emptyset, \bigcup_{\pi_i \in \pi} \pi_i = A$ )
  - Equality
  - De Morgan rules

**Relations** An  $n$ -ary relation over a set  $A$ :  $R \subseteq A^n$ .

- Binary relations  $R \subseteq A \times A \Rightarrow$  **graph representation**
  1. reflexive:  $\forall a \in A : (a, a) \in R$
  2. symmetric:  $\forall a, b \in A : (a, b) \in R \Rightarrow (b, a) \in R$
  3. antisymmetric:  $\forall a, b \in A : (a, b) \in R \Rightarrow (b, a) \notin R$
  4. transitive:  $\forall a, b, c \in A : (a, b) \in R \wedge (b, c) \in R \Rightarrow (a, c) \in R$
- 1+2+4: **equivalence**  $\Rightarrow$  partition in equivalence classes  $[a] = \{b : (a, b) \in R\}$
- 1+3+4: **partial order** (then total order)

## FUNCTIONS AND CARDINALITY

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**Functions**  $f : A \rightarrow B$ ; special relations; one-to-one, onto, bijection

- Natural isomorphism = “natural” bijection (e.g. between  $A \times B \times C$  and  $A \times (B \times C)$ , between  $A$  and  $\{\{a\} : a \in A\}$ )

**Cardinality** Binary relation (equivalence!)  $\mathcal{E}$  over the set of all sets

- $(A, B) \in \mathcal{E}$  also denoted by  $|A| = |B| \Rightarrow A$  and  $B$  are equinumerous = there exists a bijection  $e : A \rightarrow B$
- Interesting kind of sets
  - finite:  $(A, \{1, 2, \dots, n\}) \in \mathcal{E}$  for some  $n \in \mathbb{N}$ ; also written  $|A| = n$
  - (infinitely) countable:  $|A| = |\mathbb{N}|$  (count the elements)
  - uncountable
- Is  $\mathbb{N} \times \mathbb{N}$  countable?

## PROOF TECHNIQUES

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### Induction

1.  $0 \in A$ , and
2.  $\forall n : \{0, 1, \dots, n\} \subseteq A \Rightarrow n + 1 \in A$

then  $A = \mathbb{N}$

**Pigeonhole principle** If  $|A| > |B|$  then there is no one-to-one function  $f : A \rightarrow B$

- Useful example: If there is a path between vertices  $a$  and  $b$  of a graph with  $n$  vertices then there is a path between  $a$  and  $b$  of length at most  $n$

**Diagonalization** Given some relation  $R \subseteq A \times A$ , let

$$R_a = \{b : b \in A \wedge (a, b) \in R\} \quad D = \{a : a \in A \wedge (a, a) \notin R\}$$

Then  $D \neq R_a$  for any  $a \in A$

- Useful in proofs by contradiction
- Interesting examples:  $2^{\mathbb{N}}$  is uncountable;  $[0, 1]$  is uncountable

## CLOSURES

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- $R \subseteq D^{n+1}$  for some  $n > 0$ ,  $B \subseteq D$
- $B$  is **closed** under  $R$  if  $b_{n+1} \in B$  whenever  $b_1, b_2, \dots, b_n \in B$  and  $(b_1, b_2, \dots, b_n, b_{n+1}) \in R$
- **Closure property**: “ $B$  is closed under  $R_1, R_2, \dots, R_n$ ”
- Let  $\mathcal{P}$  be a closure property (under  $R_1, R_2, \dots, R_n$ ) and  $A \subseteq D$ . Then there exists a **minimal**  $B$  such that  $A \subseteq B$  and  $\mathcal{P}$  holds for  $B$ 
  - $B$  is the **closure** of  $A$  under  $R_1, R_2, \dots, R_n$
  - Useful example: The reflexive and transitive closure of  $R$  is the closure of  $R$  under reflexivity and transitivity

## ALPHABETS AND STRINGS

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- The math of strings of symbols (such as strings of bits)
- **Alphabet**  $\Sigma$ : a finite set of **symbols**
- **Strings** (not sets!) over an alphabet
- The set of all strings over  $\Sigma$ :  $\Sigma^*$
- Empty string:  $\varepsilon$  (also  $\lambda$ , in the text  $e$ )
- Operations: length ( $|w|$ ), concatenation ( $\cdot$  or juxtaposition), substring, suffix, prefix
- Length over a set  $A$ :  $|w|_A$  is the length of the string  $w$  from which all the symbols not in  $A$  have been erased
  - Abuse of notation:  $|w|_a$  is a shorthand for  $|w|_{\{a\}}$
- Exponentiation:  $w^0 = \varepsilon$ ;  $w^{i+1} = w^i w$
- Reversal:  $w = \varepsilon \Rightarrow w^{\mathbb{R}} = \varepsilon$ ; for  $a \in \Sigma$ :  $w = ua \Rightarrow w^{\mathbb{R}} = au^{\mathbb{R}}$

## LANGUAGES

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- Language: set of strings
- Can be finite, infinite, countable, etc.
- $\Sigma^*$  is a language (countable?)
- Operations: union, intersection, difference, complement ( $\bar{A} = \Sigma^* \setminus A$ )
- Concatenation:  $L_1L_2 = \{w_1w_2 : w_1 \in L_1 \wedge w_2 \in L_2\}$
- Kleene star (or closure—under what?):

$$L^* = \{w_1w_2 \cdots w_n : n \geq 0 \wedge \forall 1 \leq i \leq n : w_i \in L\}$$

- Are there languages that cannot be represented?
- We generally work with mathematical descriptions
- **Generators** are useful for describing languages
- Generally once the language is described we find convenient to work with a **regognition device** (is it the case that  $w \in L$ ?) instead

## REGULAR EXPRESSIONS AND REGULAR LANGUAGES

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- We start with very simple languages and then we combine them using a set of usual set operations
  - The set of **regular languages** is then the closure of  $\{\{a\} : a \in \Sigma\} \cup \{\emptyset\}$  under concatenation, union, and Kleene star
- Simpler to work with an inductive definition: **Regular expressions** and their associated languages are defined as follows
  - $\emptyset$  is a regular expression;  $\mathcal{L}(\emptyset) = \emptyset$
  - $a$  is a regular expression for all  $a \in \Sigma$ ;  $\mathcal{L}(a) = \{a\}$
  - If  $\alpha$  and  $\beta$  are regular expressions then so are  $\alpha\beta$ ,  $\alpha \cup \beta$ , and  $\alpha^*$ ;  
 $\mathcal{L}(\alpha\beta) = \mathcal{L}(\alpha)\mathcal{L}(\beta)$     $\mathcal{L}(\alpha \cup \beta) = \mathcal{L}(\alpha) \cup \mathcal{L}(\beta)$     $\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^*$
  - Nothing else is a regular expression
- Regular expressions are **language generators**
- The set REG of **regular languages** contain exactly all the languages generated by regular expressions