

SETS AND RELATIONS

- Sets**
- Operations: intersection, union, difference, Cartesian product
 - Big \cup , powerset (2^A), partition ($\pi \subseteq 2^A, \emptyset \notin \pi, \forall i \neq j : \pi_i \cap \pi_j = \emptyset, \bigcup_{\pi_i \in \pi} \pi_i = A$)
 - Equality
 - De Morgan rules

Relations An n -ary relation over a set A : $R \subseteq A^n$.

- Binary relations $R \subseteq A \times A \Rightarrow$ **graph representation**
 1. reflexive: $\forall a \in A : (a, a) \in R$
 2. symmetric: $\forall a, b \in A : (a, b) \in R \Rightarrow (b, a) \in R$
 3. antisymmetric: $\forall a, b \in A : (a, b) \in R \Rightarrow (b, a) \notin R$
 4. transitive: $\forall a, b, c \in A : (a, b) \in R \wedge (b, c) \in R \Rightarrow (a, c) \in R$
- 1+2+4: **equivalence** \Rightarrow partition in equivalence classes $[a] = \{b : (a, b) \in R\}$
- 1+3+4: **partial order** (then total order)

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MATHEMATICAL PRELIMINARIES/1

FUNCTIONS AND CARDINALITY

Functions $f : A \rightarrow B$; special relations; one-to-one, onto, bijection

- Natural isomorphism = "natural" bijection (e.g. between $A \times B \times C$ and $A \times (B \times C)$, between A and $\{\{a\} : a \in A\}$)

Cardinality Binary relation (equivalence!) \mathcal{E} over the set of all sets

- $(A, B) \in \mathcal{E}$ also denoted by $|A| = |B| \Rightarrow A$ and B are equinumerous = there exists a bijection $e : A \rightarrow B$
- Interesting kind of sets
 - finite: $(A, \{1, 2, \dots, n\}) \in \mathcal{E}$ for some $n \in \mathbb{N}$; also written $|A| = n$
 - (infinitely) countable: $|A| = |\mathbb{N}|$ (count the elements)
 - uncountable
- Is $\mathbb{N} \times \mathbb{N}$ countable?

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MATHEMATICAL PRELIMINARIES/2

PROOF TECHNIQUES

Induction If

1. $0 \in A$, and
2. $\forall n : \{0, 1, \dots, n\} \subseteq A \Rightarrow n + 1 \in A$

then $A = \mathbb{N}$

Pigeonhole principle If $|A| > |B|$ then there is no one-to-one function $f : A \rightarrow B$

- Useful example: If there is a path between vertices a and b of a graph with n vertices then there is a path between a and b of length at most n

Diagonalization Given some relation $R \subseteq A \times A$, let

$$R_a = \{b : b \in A \wedge (a, b) \in R\} \quad D = \{a : a \in A \wedge (a, a) \notin R\}$$

Then $D \neq R_a$ for any $a \in A$

- Useful in proofs by contradiction
- Interesting examples: $2^{\mathbb{N}}$ is uncountable; $[0, 1]$ is uncountable

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MATHEMATICAL PRELIMINARIES/3

CLOSURES

- $R \subseteq D^{n+1}$ for some $n > 0, B \subseteq D$
- B is **closed** under R if $b_{n+1} \in B$ whenever $b_1, b_2, \dots, b_n \in B$ and $(b_1, b_2, \dots, b_n, b_{n+1}) \in R$
- **Closure property**: " B is closed under R_1, R_2, \dots, R_n "
- Let \mathcal{P} be a closure property (under R_1, R_2, \dots, R_n) and $A \subseteq D$. Then there exists a **minimal** B such that $A \subseteq B$ and \mathcal{P} holds for B
 - B is the **closure** of A under R_1, R_2, \dots, R_n
 - Useful example: The reflexive and transitive closure of R is the closure of R under reflexivity and transitivity

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MATHEMATICAL PRELIMINARIES/4

- The math of strings of symbols (such as strings of bits)
- **Alphabet** Σ : a finite set of **symbols**
- **Strings** (not sets!) over an alphabet
- The set of all strings over Σ : Σ^*
- Empty string: ε (also λ , in the text e)
- Operations: length ($|w|$), concatenation (\cdot or juxtaposition), substring, suffix, prefix
- Length over a set A : $|w|_A$ is the length of the string w from which all the symbols not in A have been erased
 - Abuse of notation: $|w|_a$ is a shorthand for $|w|_{\{a\}}$
- Exponentiation: $w^0 = \varepsilon$; $w^{i+1} = w^i w$
- Reversal: $w = \varepsilon \Rightarrow w^R = \varepsilon$; for $a \in \Sigma$: $w = ua \Rightarrow w^R = aw^R$

- Language: set of strings
- Can be finite, infinite, countable, etc.
- Σ^* is a language (countable?)
- Operations: union, intersection, difference, complement ($\bar{A} = \Sigma^* \setminus A$)
- Concatenation: $L_1 L_2 = \{w_1 w_2 : w_1 \in L_1 \wedge w_2 \in L_2\}$
- Kleene star (or closure—under what?):

$$L^* = \{w_1 w_2 \cdots w_n : n \geq 0 \wedge \forall 1 \leq i \leq n : w_i \in L\}$$

- Are there languages that cannot be represented?
- We generally work with mathematical descriptions
- **Generators** are useful for describing languages
- Generally once the language is described we find convenient to work with a **regognition device** (is it the case that $w \in L$?) instead

REGULAR EXPRESSIONS AND REGULAR LANGUAGES

- We start with very simple languages and then we combine them using a set of usual set operations
 - The set of **regular languages** is then the closure of $\{\{a\} : a \in \Sigma\} \cup \{\emptyset\}$ under concatenation, union, and Kleene star
- Simpler to work with an inductive definition: **Regular expressions** and their associated languages are defined as follows
 - \emptyset is a regular expression; $\mathcal{L}(\emptyset) = \emptyset$
 - a is a regular expression for all $a \in \Sigma$; $\mathcal{L}(a) = \{a\}$
 - If α and β are regular expressions then so are $\alpha\beta$, $\alpha \cup \beta$, and α^* ;

$$\mathcal{L}(\alpha\beta) = \mathcal{L}(\alpha)\mathcal{L}(\beta) \quad \mathcal{L}(\alpha \cup \beta) = \mathcal{L}(\alpha) \cup \mathcal{L}(\beta) \quad \mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^*$$
 - Nothing else is a regular expression
- Regular expressions are **language generators**
- The set REG of **regular languages** contain exactly all the languages generated by regular expressions