RADICALS OF EXTENDED SMASH PRODUCTS OF GROUP-GRADED RINGS

by

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A Note on this Thesis

I deleted the original copy of the DVI file and had to recompile this without the aid of the Dalhousie thesis style. I substituted instead the equivalent file from the University of New Brunswick, where I am studying for a PhD.

Any typographical errors in this copy are due to the difference in style files.

Dedication

To
the Alpha, the Omega, the beginning and end,
the God of all creation, and my personal friend;
and to
Lady Mekan, for giving me a reason to finish.

Acknowledgements

Praise be to the God and Father of my LORD Jesus Christ, who has blessed me in the heavenly realms, and has given me the insight to finish this thesis. For without Him, none of this would have been written.

I thank my parents for knowing when to prod me (which was all the time), and for their prayers. I also thank my friends and roommates at the "Yale Street Hotel" for their continued prayer and encouragement, as well as those of my friends at the "$\Delta\iota\alpha\kappa o\nu\alpha\tau\epsilon$ Mission".

A very special thanks to the Lady mentioned in the dedication for her encouragement and prayers and offers of food (important to any student).

I am deeply in the debt of my supervisor, who was patient with me above and beyond the call of duty. I thank her for her guidance, in showing me the way to go, but not taking me there. As well, I thank Dr. Stewart for his invaluable advise.

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Abstract

In this thesis we take a group graded ring, R, that has unity, and show two equivalent conditions for it to be graded. Then, using the dual of the group ring, $k[G]$, denoted $k[G]^*$, and a subalgebra of it called P_G, we form a new ring called the smash product, written $R\#P_G$. We do this for both finite and infinite groups. In the infinite case, the smash product has no unity, so we adjoin a 1 using a method similar to the Dorroh extension.

Next we introduce the ideas of radicals of rings with some examples. Talking about graded rings, we introduce graded radicals. With this theory, we are able to characterize certain radicals of the smash product. Saorin gave a characterization of the smash product with a 1 adjoined for the Jacobson radical. Here we will do so for all hereditary radicals, thus making Saorin's result a corollary.

Chapter 1
Preliminaries

In this thesis, we examine radicals of group graded rings, and their relationship to the radical of the smash product and the radical of the smash product with a 1 adjoined.

We start by first reviewing elementary terms in ring theory, and from there move on to defining the smash product of a graded ring.

Throughout this thesis, we will be considering rings over a commutative ring with unity, denoted k. Since we are working over a commutative ring with unity, k, we require our ring, R, to be a left and right k-module with the left and right actions being the same. This is possible, since at the very least, every ring is a $\mathbf{Z}$-module. All maps are assumed to be k-linear, unless it is otherwise specified.

1.1 Modules and Algebras

Definition 1.1.1 *Let R be a ring. We say M is a left R-module if M is an abelian group under addition and there is a map from $R \times M \longrightarrow M$ denoted $(r, m) \longmapsto rm$ such that for all $r, s \in R$ and $m, n \in M$ we have*

1. $r(m+n) = rm + rn$
2. $(r+s)m = rm + sm$
3. $(rs)m = r(sm)$
4. $RM = M$

A right R-module is defined in a similar manner. We sometimes say that M is a module over the ring R. Unless specified otherwise, all modules will be considered to be left modules. We also assume that all modules, M, are left and right k-modules with $lm = ml$ for all $m \in M$ and $l \in k$.

Now if M is an R-module, and there is a set of elements, $\{m_i \in M | i \in I\}$, such that every element $m \in M$ can be written as unique sum of the form $m = \sum_{i \in I} r_i m_i$, where $r_i \in R$, and only a finite number of the r_i's are non-zero, we say that M is a free R-module, and we call the set of elements a basis for the module.

Example 1.1.2 For any ring R, the matrix ring $M^*_G(R)$ is a left R-module. $M^*_G(R)$ denotes the ring of matrices indexed by some group, G, with a finite number of entries from the ring R. The action of R on this module is as follows: for all $r \in R$ and all $A \in M^*_G(R)$

$$(rA)(g,h) = r(A(g,h))$$

where g, h are elements of G. This action satisfies the conditions of Definition 1.1.1. We can also define the action of R on the right of $M^*_G(R)$ similarly. Thus $M^*_G(R)$ is both a left and right R-module. As well, $M^*_G(R)$ is a free R-module with a basis $\{e(g,h) | g, h \in G\}$, where $e(g,h)$ is the matrix with a 1 in the $(g,h)^{th}$ position and zeros elsewhere. †

Definition 1.1.3 *If M is both a left R-module and a right S-module, and satisfies the condition that $(rm)s = r(ms)$ for all $r \in R$, $s \in S$ and $m \in M$, then we say that M is an $R-S$-bimodule. If $R = S$, then we abbreviate by saying that M is an R-bimodule.*

Let M be an R-module, and let $S_i, i \in I$ be submodules. Suppose each $m \in M$ can be written uniquely as a finite sum of elements, one from each of the S_i. Then we say that M is the direct sum of $S_i, i \in I$, and write this as $M = \bigoplus_{i \in I} S_i$. We can write each element $m \in M$ as $m = s_{i_1} + s_{i_2} + s_{i_3} + \cdots$ where $s_{i_j} \in S_{i_j}$, and only finitely many of them are non-zero. By the uniqueness condition above, the S_i are disjoint except for the 0 element.

Definition 1.1.4 *Let A be a right R-module and B a left R-module. The tensor product, $A \otimes_R B$, is the free k-module on the abelian group $A \times B$ where $A \times B$ is the Cartesian product mod H, and H is the subgroup of $A \times B$ generated by elements of the form*

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b)$$

$$(a, b_1 + b_2) - (a, b_1) - (a, b_2)$$

$$(ar, b) - (a, rb)$$

for all $a_i \in A$, $b_i \in B$, and $r \in R$. The class of $(a,b) \in A \times B$ in this quotient is denoted $a \otimes_R b$ and a general element of $A \otimes_R B$ is a finite sum of such elements. If $k \subseteq R$, the tensor product, $A \otimes_R B$, is a right and left k-module by

$$l(a \bigotimes_R b) = la \bigotimes_R b = al \bigotimes_R b = a \bigotimes_R lb = a \bigotimes_R bl = (a \bigotimes_R b)l.$$

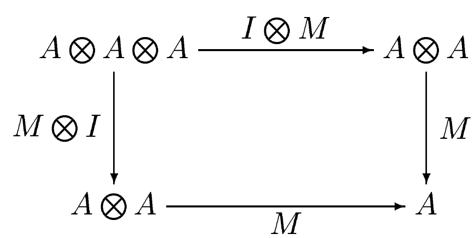
The tensor $\otimes$ without subscript will mean $\otimes_k$.

Proposition 1.1.5 *For a right k-module A, $A \bigotimes k \cong A$. For a left k-module A, $k \bigotimes A \cong A$.*

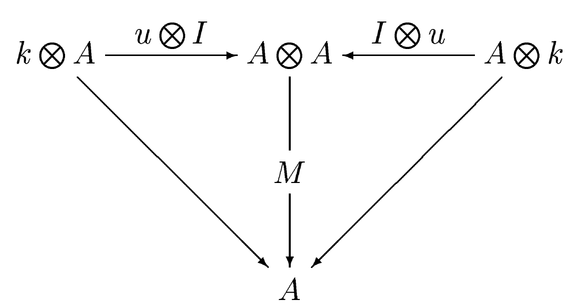
Proof: Let ϕ map $A \times k$ to A by $\phi(\sum(a_i, l_i)) = \sum a_i l_i$. Since ϕ clearly maps any element of the subgroup H of $A \times k$ from Definition 1.1.4 to 0, ϕ defines a map from $A \bigotimes k$ to A. Let $\eta : A \longrightarrow A \bigotimes k$ be defined by $\eta(a) = a \bigotimes 1$. Then $\eta\phi(\sum a_i \bigotimes l_i) = \eta(\sum a_i l_i) = \sum a_i l_i \bigotimes 1 = \sum a_i \bigotimes l_i$, and $\phi\eta(a) = \phi(a \bigotimes 1) = a$, so $A \bigotimes k \cong A$. Similarly, $k \bigotimes A \cong A$. □

Let us now use modules to define a new object. To start, let A be a k-module together with two maps, multiplication $M : A \bigotimes A \longrightarrow A$ and the unit map $u : k \longrightarrow A$, such that the following diagrams commute:

1. Associativity



2. Unitary Property



($k \bigotimes A \longrightarrow A$ and $A \bigotimes k \longrightarrow A$ are the isomorphisms given in Proposition 1.1.5.)

Definition 1.1.6 *We say that a k-module, A, is an k-algebra if it satisfies the above conditions. The algebra is sometimes written as (A, M, u), or if more than one algebra are being considered, (A, M_A, u_A). Unless otherwise stated, an algebra is considered to be a k-algebra.*

Definition 1.1.7 *If A and B are algebras, and $f : A \longrightarrow B$ is a k-linear map, then f is an algebra map if*

1. $f(ab) = f(a)f(b)$

2. $f(u_A(1_k)) = u_B(1_k)$

Recall that we are assuming that all rings R are k-modules.

Proposition 1.1.8 *If R is a ring with unity, then R is a k-algebra.*

Proof: Let us define the map M to be ring multiplication, and let $u(n) = n1_R$ for any $n \in k$, using module multiplication. Then the associative property holds because ring multiplication is associative. For the unitary property, note that $M(u \bigotimes I)(n \bigotimes r) = nr$, which is the result of the maps $k \bigotimes R \longrightarrow R$ and $R \bigotimes k \longrightarrow R$. Therefore R is a k-algebra. □

Proposition 1.1.9 *Let A and B be k-algebras and let $M_{A \bigotimes B} : A \bigotimes B \bigotimes A \bigotimes B \longrightarrow A \bigotimes B$ be defined by $M_{A \bigotimes B}(a \bigotimes b \bigotimes c \bigotimes d) = ac \bigotimes bd$, and extended by linearity, and let $u_{A \bigotimes B} : k \longrightarrow A \bigotimes B$ be defined by $u_{A \bigotimes B}(l) = lu_A(1) \bigotimes u_B(1)$. Then $A \bigotimes B$ is a k-algebra.*

Proof: For associativity,

$$\begin{aligned}(a \bigotimes b)((c \bigotimes d)(e \bigotimes f)) &= (a \bigotimes b)(ce \bigotimes df) \\ &= a(ce) \bigotimes b(df) \\ &= (ac)e \bigotimes (bd)f \\ &= (ac \bigotimes bd)(e \bigotimes f) \\ &= ((a \bigotimes b)(c \bigotimes d))(e \bigotimes f)\end{aligned}$$

and the unitary property follows directly from the definition of $u_{A \bigotimes B}$. So the tensor product of two k-algebras is again a k-algebra. □

1.2 Group-Graded Rings

Definition 1.2.1 *We say R is a G-graded ring, where G is a group, if there is a family of additive subgroups $\{R_g | g \in G\}$ of R such that*

1. $R = \bigoplus_{g \in G} R_g$.

2. $R_g R_h \subseteq R_{gh}$ *for all* $g, h \in G$

The elements of R_g are denoted r_g, and are called homogeneous elements of R. A ring R is called strongly G-graded if $R_g R_h = R_{gh}$ for all $g, h \in G$.

A subring, S, of a G-graded ring R, with $S = \bigoplus (S \cap R_g)$ is a G-graded subring of R.

Example 1.2.2 For any ring R, we can consider the following two gradings. First, we can grade R by the trivial group $\{e\}$. The other grading, called the trivial grading of R by a group G is as follows: $R_e = R$ and $R_g = 0$ for all $g \neq e$. The trivial grading does not depend on the group, so for any group, G, and any ring, R, we can consider R to be a G-graded ring. †

Proposition 1.2.3 *Let R be a G-graded ring with unity. Then the following are equivalent.*

1. *$R_g R_h = R_{gh}$ for all $g, h \in G$; that is, R is strongly graded.*
2. *$R_g R_{g^{-1}} = R_e$ for all $g \in G$, where e is the identity in G*

Proof: That the first implies the second is clear. The reverse implication is not so obvious. Since R is graded, we have $R_g R_h \subseteq R_{gh}$, so all that is required to show is that each $r_{gh} \in R_{gh}$ is contained in $R_g R_h$. Now the $1 \in R$ must lie in $R_e = R_g R_{g^{-1}}$, since $1 r_g = r_g$, and if 1 was not in R_e, this could not happen. Thus we know that $1 = \sum_{i=1}^{n} s(i)t(i)$ for some $s(i) \in R_g$ and some $t(i) \in R_{g^{-1}}$. Now, let us multiply this on the right by r_{gh}. This gives $\sum_{i=1}^{n} s(i)t(i)r_{gh}$. Since $t(i)r_{gh} \in R_h$ for all i, the statement is proved. □

Example 1.2.4 Let R be a ring, and let $R[x]$ denote the ring of polynomials. In this ring, the operations are defined as follows:

$$\sum_{i=0}^{m} r_i x^i + \sum_{j=0}^{n} s_j x^j = \sum_{i=0}^{\max(m,n)} (r_i + s_i) x^i \tag{1.1}$$

and

$$\sum_{i=0}^{m} r_i x^i \sum_{j=0}^{n} s_j x^j = \sum_{z=0}^{m+n} \sum_{i=0}^{z} (r_i s_{z-i}) x^z \tag{1.2}$$

For any ring R, $R[x]$ is graded by $\mathbf{Z}$. The grading is as follows: for $n < 0$, $R[x]_n = \{0\}$, and for $n \geq 0$, $R[x]_n = \{rx^n | r \in R\}$. A homogeneous element of $R[x]$ looks like rx^n. For example, in $\mathbf{Z}[x]$, $5x^2$ is homogeneous but $x^2 + 2x + 1$ is not. Note that $R[x]$ is not strongly graded, since $R_n R_{-n} = 0$ for all non-zero n. †

Example 1.2.5 Another example of a graded ring is the group ring. Let R be any ring and let G be any group. Then the group ring, denoted $R[G]$, is the free left R-module with basis $\{u_g | g \in G\}$ and multiplication defined by $(ru_g)(su_h) = rsu_{gh}$. We usually write g to denote the basis element u_g. In other wo rds, we can write the group ring as follows:

$$R[G] = \bigoplus_{g\in G} Rg = \{\sum_{g\in G} r(g)g | r(g) \in R\}.$$

The binary operations are given as

$$\sum_{g\in G} r(g)g + \sum_{h\in G} s(h)h = \sum_{g\in G}(r(g) + s(g))g \tag{1.3}$$

and

$$\sum_{g\in G} r(g)g \sum_{h\in G} s(h)h = \sum_{g\in G}(\sum_{ab=g}(r(a)s(b)))g \tag{1.4}$$

It is easy to see that $R[G]$ is a G-graded ring, with $R[G]_g = Rg$. Note that $R[G] \cong R \bigotimes k[G]$ if R is a k-algebra.

If either the ring, R, or the group, G, is not commutative, the group ring will not be commutative. If R is a ring with unity, then the group ring has unity, with $1_{R[G]} = 1_R e$. Thus by Proposition 1.1.8, the group ring would be a k-algebra. Let M and u be the k-linear maps defined by $M(g \bigotimes h) = (gh)$ and $u(1) = e$ for $g, h \in G$.

Note that the polynomial ring, $R[x]$ is isomorphic to a subring of $R[\mathbf{Z}]$. †

Proposition 1.2.6 *If $RR = R$, $R[G]$ is strongly G-graded.*

Proof: In Example 1.2.5, we stated that $R[G]$ is G-graded. Thus, all that we are required to show is $R(gh) \subseteq RgRh$, since inclusion in the other direction is given by the grading. Let $r(gh) \in R(gh)$. Since $RR = R$, there are elements $e_i, f_i \in R$ such that $\sum e_i f_i = r$. Thus we have that $\sum(e_i g)(f_i h) = r(gh)$. □

Definition 1.2.7 *Let A be a k-module. Then the set of all k-linear module homomorphisms from A to k forms a k-module. For two homomorphism ϕ, ψ, $\phi + \psi(a) = \phi(a) + \psi(a)$, and $l\phi(a) = \phi(la)$, for all $a \in A$ and $l \in k$. This module is denoted A^*, or $Hom_k(A, k)$, and is called the dual of A.*

Example 1.2.8 Let G be a group and $k[G]$ be the group ring, which is a k-algebra. The dual of $k[G]$, which is denoted $k[G]^*$, is also a k-algebra. Define the multiplication map $M_{k[G]^*}$ as $M_{k[G]^*}(\phi \bigotimes \psi)(g) = \phi(g)\psi(g)$, where $\phi, \psi \in k[G]^*$ and $g \in G$.

This map is extended by linearity to finite sums in $k[G]$. For the unit map, let $u_{k[G]^*}(n)(\sum_{g\in G} m(g)g) = n\sum_{g\in G} m(g)$.

Now multiplication is clearly associative since the ring k is associative. The unitary property is true from the definition of the unit map. Therefore $k[G]^*$ is a k-algebra.

We consider the subalgebra of $k[G]^*$ generated by the projection maps p_g, where $p_h(\sum_{g\in G} n(g)g) = n(h)$. We will denote this subring as P_G. For a finite group, $P_G = k[G]^*$ and has multiplicative identity $\sum_{g\in G} p_g$. If G is infinite, the inclusion $P_G \subset k[G]^*$ is proper and P_G has no multiplicative identity.

The subalgebra P_G will play a key role in later chapters. †

Definition 1.2.9 *Let R be a ring, and G be a group. We say that G acts as a group of automorphisms on R if there is a homomorphism from G into the group of ring automorphisms of R.*

Example 1.2.10 Let $\mathbf{R}$ be the set of real numbers. Then $\mathbf{R}^4$ is a ring with addition and multiplication performed component-wise; that is, $(r_1, r_2, r_3, r_4)+(s_1, s_2, s_3, s_4) = (r_1+s_1, r_2+s_2, r_3+s_3, r_4+s_4)$, and $(r_1, r_2, r_3, r_4)(s_1, s_2, s_3, s_4) = (r_1s_1, r_2s_2, r_3s_3, r_4s_4)$. $\mathbf{R}^4$ is the set of linear combinations of $\{e_0, e_1, e_2, e_3\}$ with coefficients in $\mathbf{R}$, where e_i has 1 in the i^{th} position and 0 elsewhere. Consider the maps $\phi_i : \mathbf{R}^4 \longrightarrow \mathbf{R}^4$, $i = 0, 1, 2, 3$ given by $\phi_i(e_j) = e_{(i+j)}$ with the index addition performed in $\mathbf{Z}_4$. It is easily shown that these maps are ring automorphisms.

These maps form a group under composition. This group is isomorphic to $\mathbf{Z}_4$. So we can think of $\mathbf{Z}_4$ acting on $\mathbf{R}^4$ as a group of automorphisms. †

Example 1.2.11 If G acts on R as a group of automorphisms, we can consider another algebraic structure similar to the group ring. By slightly altering the multiplication rule for the group ring, we can form what is known as a skew group ring. The new multiplication is defined by $(rg)(sh) = (rg(s))(gh)$. This multiplication is associative, as will be shown:

$$\begin{aligned}
(rg)((sh)(ti)) &= (rg)((sh(t))(hi)) \\
&= (rg(sh(t)))(ghi) \\
&= (rg(s)g(h(t)))(ghi) \\
&= (rg(s))(gh)(ti) \\
&= ((rg)(sh))(ti)
\end{aligned}$$

†

1.3 Ideals

An ideal, I, of a ring, R, is a subring with the additional property that for any $r \in R$, $rI \subseteq I$ and $Ir \subseteq I$. If only one of the inclusions hold, then we say that I is a left, or right, ideal, depending on which side the elements of R are multiplied on. Throughout this discussion, ideals will be two-sided unless otherwise stated. Two-sided ideals are the kernels of ring homomorphisms.

We say an ideal, I, is graded if I is a graded subring of the ring, R, $R_g I_h \subseteq I_{gh}$ and $I_h R_g \subseteq I_{hg}$.

Example 1.3.1 An ideal, I, of a graded ring, R, may not be a graded subring. For example, consider the ring of polynomials, $R = k[x]$ from Example 1.2.4 and the ideal $I = \langle x+1 \rangle = \{f(x)(x+1) | f(x) \in k[x]\}$. The ring, R, is graded by the integers, but the ideal, I, is not a graded subring since $x+1$ is in I but neither x nor 1 is. For such ideals, I, we can consider the largest graded ideal that is contained in I. This is equal to $\bigoplus_{g \in G} I \cap R_g$, and is denoted I_G. In this case, $\langle x+1 \rangle_{\mathbf{Z}} = 0$. †

Definition 1.3.2 *We say an ideal, $I \neq R$, is maximal in R, if for any ideal J such that $I \subseteq J$ and $I \neq J$, then $J = R$.*

Definition 1.3.3 *The intersection of all ideals of R which contain a given nonempty set of elements K, is called the ideal generated by K, and is denoted (K) [12, Definition 2.3].*

Proposition 1.3.4 *For an ideal P of a ring R the following are equivalent:*

1. *if A, B are ideals of R and $AB \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$.*
2. *if $aRb \subseteq P$ then either $a \in P$ or $b \in P$.*

Proof: (**1** $\Rightarrow$ **2**) Suppose $aRb \subseteq P$, for $a, b \in R$. Then it is clear that $(RaR)(RbR) \subseteq P$. Since both RaR and RbR are ideals of R, by the assumption of 1), we have that one of them is contained in P. Assume it is RaR. Take $A = (a)$ which is equal to $\{na + s_0 a + a t_0 + \sum_{i=1}^{n} s_i a t_i | s_i, t_i \in R, \quad n \in \mathbf{Z}\}$ [12, 2.6]. Then $AAA \subseteq RaR \subseteq P$. Again, by the assumption of 1), we have that $A \subseteq P$, hence $a \in P$.

(**2** $\Rightarrow$ **1**) Suppose that $AB \subseteq P$, for ideals A and B. Then we have that $ARB \subseteq P$. Now suppose that neither A nor B are contained in P. That means that there is an $a \in A$ and a $b \in B$ such that a and b are not in P. But $aRb \subseteq ARB \subseteq P$. So by the assumption of 2), one of a or b has to be in P. This contradicts the assumption that neither A nor B is contained in P. Thus 1) is shown. □

Definition 1.3.5 *We say an ideal, P, is prime if it satisfies either of the above conditions.*

Proposition 1.3.6 *Let R be a commutative ring with unity. Then M is a maximal ideal if and only if R/M is a field [8, Theorem 5.9].*

Proposition 1.3.7 *Let R be a commutative ring with unity, and let $P \neq R$ be an ideal in R. Then P is a prime ideal if and only if R/P is an integral domain; that is, it contains no divisors of* 0 *[8, Theorem 5.10].*

Example 1.3.8 Let us consider the ring $R = \mathbf{Z} \times \mathbf{Z}$ with addition and multiplication performed component-wise. It is easy to see that R is a commutative ring with unity. We will take an ideal of the ring R which is prime, but not maximal. Let $I = 0 \times \mathbf{Z}$. Then R/I is isomorphic to $\mathbf{Z}$, which is indeed an integral domain, however it is not a field as it lacks inverses. Hence, by Propositions 1.3.6 and 1.3.7, we see that I is a prime ideal, but not maximal. †

Chapter 2

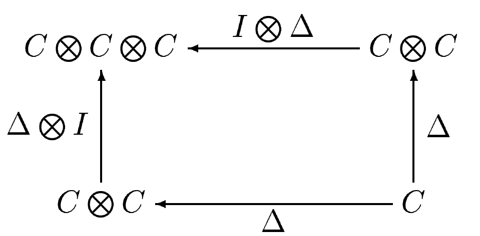
Group Rings and Graded Rings

In this chapter we will discuss the relationship between a G-graded ring R and the group ring $k[G]$. As well, if G is a finite group, we will note a relationship between a G-graded ring, R, and $k[G]^*$. However, we will begin by exploring the structure of $k[G]$ and $k[G]^*$ further.

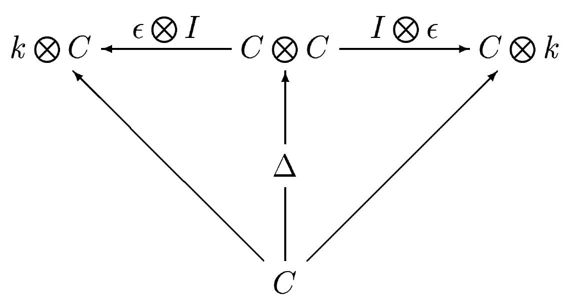
2.1 Bialgebras

Let C be a k-module together with two k-module homomorphisms, comultiplication $\Delta : C \longrightarrow C \otimes C$ and the counit map $\epsilon : C \longrightarrow k$, such that the following diagrams commute:

1. Coassociativity



2. Counitary Property



Definition 2.1.1 *If a k-module, C, satisfies the above conditions, then we say that C is a k-coalgebra. The coalgebra is sometimes written as (C, Δ, ϵ), or if more than one*

coalgebra are being considered, $(C, \Delta_C, \epsilon_C)$. *Unless specified otherwise, a coalgebra is considered to be a* k*-coalgebra.*

The definition of the coalgebra is dual to that of an algebra in the sense that the arrows in the diagrams are "reversed". For a general coalgebra, C, we denote $\Delta_C(c)$ as $\sum_{(c)} c_{(1)} \otimes c_{(2)}$. In the same way, we define $(\Delta_C \otimes I)\Delta_C(c)$ to be $\sum_{(c)} c^{(1)}_{(1)} \otimes c^{(2)}_{(1)} \otimes c_{(2)}$, and $(I \otimes \Delta_C)\Delta_C(c) = \sum_{(c)} c_{(1)} \otimes c^{(1)}_{(2)} \otimes c^{(2)}_{(2)}$.

Example 2.1.2 Let C and D be k-coalgebras, and form the tensor product $C \otimes D$.

Now, let us define maps $\Delta_{C \bigotimes D} : C \otimes D \longrightarrow C \otimes D \otimes C \otimes D$ and $\epsilon_{C \bigotimes D} : C \otimes D \longrightarrow k$ as follows:

$$\Delta_{C \bigotimes D}(c \bigotimes d) = (I_C \bigotimes T \bigotimes I_D)(\Delta_C(c) \bigotimes \Delta_D(d)), \tag{2.1}$$

where T is the twist map, and

$$\epsilon_{C \bigotimes D}(c \bigotimes d) = \epsilon_C(c)\epsilon_D(d). \tag{2.2}$$

We claim that $(C \otimes D, \Delta_{C \bigotimes D}, \epsilon_{C \bigotimes D})$ is a coalgebra. Let us check that $\Delta_{C \bigotimes D}$ satisfies the coassociativity property. Before we begin, we define the map T_{ij} to be map which switches the elements in the i^{th} and j^{th} positions.

Now let $c \in C$ and $d \in D$. Then

$$\begin{array}{rcl} (\Delta_{C \bigotimes D} \otimes I)\Delta_{C \bigotimes D}(c \otimes d) & = & \sum_{(c),(d)} c^{(1)}_{(1)} \otimes d^{(1)}_{(1)} \otimes c^{(2)}_{(1)} \otimes d^{(2)}_{(1)} \otimes c_{(2)} \otimes d_{(2)} \\ & \xrightarrow{T_{23}} & \sum_{(c),(d)} c^{(1)}_{(1)} \otimes c^{(2)}_{(1)} \otimes d^{(1)}_{(1)} \otimes d^{(2)}_{(1)} \otimes c_{(2)} \otimes d_{(2)} \\ & \xrightarrow{T_{45}} & \sum_{(c),(d)} c^{(1)}_{(1)} \otimes c^{(2)}_{(1)} \otimes d^{(1)}_{(1)} \otimes c_{(2)} \otimes d^{(2)}_{(1)} \otimes d_{(2)} \\ & \xrightarrow{T_{34}} & \sum_{(c),(d)} c^{(1)}_{(1)} \otimes c^{(2)}_{(1)} \otimes c_{(2)} \otimes d^{(1)}_{(1)} \otimes d^{(2)}_{(1)} \otimes d_{(2)} \\ & \overset{coass.}{=} & \sum_{(c),(d)} c_{(1)} \otimes c^{(1)}_{(2)} \otimes c^{(2)}_{(2)} \otimes d_{(1)} \otimes d^{(1)}_{(2)} \otimes d^{(2)}_{(2)} \\ & \xrightarrow{T_{34}} & \sum_{(c),(d)} c_{(1)} \otimes c^{(1)}_{(2)} \otimes d_{(1)} \otimes c^{(2)}_{(2)} \otimes d^{(1)}_{(2)} \otimes d^{(2)}_{(2)} \\ & \xrightarrow{T_{45}} & \sum_{(c),(d)} c_{(1)} \otimes c^{(1)}_{(2)} \otimes d_{(1)} \otimes d^{(1)}_{(2)} \otimes c^{(2)}_{(2)} \otimes d^{(2)}_{(2)} \\ & \xrightarrow{T_{23}} & \sum_{(c),(d)} c_{(1)} \otimes d_{(1)} \otimes c^{(1)}_{(2)} \otimes d^{(1)}_{(2)} \otimes c^{(2)}_{(2)} \otimes d^{(2)}_{(2)} \\ & = & (I \otimes \Delta_{C \bigotimes D})\Delta_{C \bigotimes D}(c \otimes d). \end{array}$$

Now we have a string of twist maps with an equality in the middle. We will show that the string of maps is the identity, thereby proving that $\Delta_{C \bigotimes D}$ is coassociative.

First, note that the inverse of T_{ij} is T_{ij}. Then, we see that the above maps are in this order:

$$T_{23}T_{45}T_{34}T_{34}T_{45}T_{23}$$

This is clearly the identity, so equality is shown.

To show that $C \bigotimes D$ is counitary, let $c \in C$ and $d \in D$. Then

$$\begin{array}{rcl}(\epsilon_{C \bigotimes D} \bigotimes I)\Delta_{C \bigotimes D}(c \bigotimes d) & = & \sum_{(c),(d)} \epsilon_C(c_{(1)})\epsilon_D(d_{(1)}) \bigotimes c_{(2)} \bigotimes d_{(2)} \\ & = & \sum_{(c),(d)} 1_k \bigotimes \epsilon_C(c_{(1)})c_{(2)} \bigotimes \epsilon_D(d_{(1)})d_{(2)} \\ & \overset{counitary}{=} & 1_k \bigotimes c \bigotimes d \\ & = & c \bigotimes d\end{array}$$

The other part of the counitary diagram is shown similarly.

Therefore $(C \bigotimes D, \Delta_{C \bigotimes D}, \epsilon_{C \bigotimes D})$ is a coalgebra. So we have shown that the tensor product of two coalgebras is again a coalgebra. †

Example 2.1.3 With the k-linear maps, Δ and ϵ, defined by $\Delta(g) = g \bigotimes g$ and $\epsilon(g) = 1$ for all $g \in G$, $k[G]$ is a coalgebra. Clearly Δ is coassociative. To show that the group ring is counitary, we note that $(\epsilon \bigotimes I)(\Delta(g)) = (\epsilon \bigotimes I)(g \bigotimes g) = 1_k \bigotimes g$. This is also the result given by the isomorphism defined in Proposition 1.1.5. †

Example 2.1.4 Let G be a finite group, and let $k[G]$ be the group ring. Define $p_g \in k[G]^*$ as in Example 1.2.8. As mentioned in Example 1.2.5, $k[G]$ is generated as a k-module by G. For G finite, $k[G]^*$ is generated as a k-module by the p_g's.

Now let $\Delta : k[G]^* \longrightarrow k[G]^* \bigotimes k[G]^*$ be given by $\Delta(p_g) = \sum_{h \in G} p_h \bigotimes p_{h^{-1}g}$ and let $\epsilon(p_g) = 1_k$ if $g = e$ and 0 otherwise, where ϵ goes from $k[G]^*$ to k.

We claim that $(k[G]^*, \Delta, \epsilon)$ is a coalgebra. For the comultiplication, we observe that

$$\begin{array}{rcl}(I \bigotimes \Delta)\Delta(p_g) & = & (I \bigotimes \Delta)(\sum_{h \in G} p_h \bigotimes p_{h^{-1}g}) \\ & = & \sum_{h \in G} p_h \bigotimes \sum_{j \in G} p_j \bigotimes p_{j^{-1}h^{-1}g} \\ & = & \sum_{h,l \in G} p_h \bigotimes p_{h^{-1}l} \bigotimes p_{l^{-1}g} \\ & = & (\Delta \bigotimes I)(\sum_{l \in G} p_l \bigotimes p_{l^{-1}g}) \\ & = & (\Delta \bigotimes I)\Delta(p_g)\end{array}$$

As for counitary, we have the following:

$$\begin{aligned}(\epsilon \otimes I)\Delta(p_g) &= (\epsilon \otimes I)(\textstyle\sum_{h\in G} p_h \otimes p_{h^{-1}g}) \\ &= 1 \otimes p_{e^{-1}g} \\ &= 1 \otimes p_g\end{aligned}$$

$$\begin{aligned}(I \otimes \epsilon)\Delta(p_g) &= (I \otimes \epsilon)(\textstyle\sum_{h\in G} p_h \otimes p_{h^{-1}g}) \\ &= p_{ge^{-1}} \otimes 1 \\ &= p_g \otimes 1\end{aligned}$$

Therefore we have shown that $(k[G]^*, \Delta, \epsilon)$ is a coalgebra. †

Definition 2.1.5 *If A and B are coalgebras and $f : A \longrightarrow B$ is a k-linear map, then f is a coalgebra map if*

1. $(f \otimes f)(\Delta_A(a)) = \Delta_B(f(a))$
2. $\epsilon_B(f(a)) = \epsilon_A(a)$

Definition 2.1.6 *Let (H, M, u) be an algebra and (H, Δ, ϵ) be a coalgebra. Then we say $(H, M, u, \Delta, \epsilon)$ is a bialgebra if either of the following two equivalent conditions are satisfied.*

1. *M and u are coalgebra maps, or*
2. *Δ and ϵ are algebra maps.*

Let us note that k is an algebra, and a coalgebra with M_k being the normal ring multiplication, $\Delta_k(l) = 1 \otimes l$, and both u_k and ϵ_k being the identity map.

Proposition 2.1.7 *For a group G, $k[G]$ is a bialgebra.*

Proof: From Examples 1.2.5 and 2.1.3 we know that $k[G]$ is both an algebra and a coalgebra. To show that $k[G]$ is a bialgebra, we will show that M and u are coalgebra maps.

First, from the following, we shall see that M is a coalgebra map. Let $g, h \in k[G]$ and $l \in k$. Then

$$
\begin{aligned}
1.\quad (M\otimes M)(\Delta_{k[G]\otimes k[G]}(g\otimes h)) &= (M\otimes M)(g\otimes h\otimes g\otimes h)\\
&= gh\otimes gh\\
&= \Delta(gh)\\
&= \Delta(M(g\otimes h))
\end{aligned}
$$

$$
\begin{aligned}
2.\quad \epsilon(M(g\otimes h)) &= \epsilon(gh)\\
&= 1\\
&= \epsilon(g)\epsilon(h)\\
&= \epsilon_{k[G]\otimes k[G]}(g\otimes h)
\end{aligned}
$$

Next, we will show that u is a coalgebra map. Let $l\in k$. Then

$$
\begin{aligned}
1.\quad (u\otimes u)(\Delta_k(l)) &= (u\otimes u)(l\otimes 1)\\
&= le\otimes e\\
&= \Delta(le)\\
&= \Delta(u(l))
\end{aligned}
$$

$$
\begin{aligned}
2.\quad \epsilon(u(l)) &= \epsilon(le)\\
&= l\\
&= \epsilon_k(l)
\end{aligned}
$$

Thus we see that M and u are coalgebra maps. So, by Definition 2.1.6, $k[G]$ is a bialgebra. □

Proposition 2.1.8 *Let G be a finite group. Then $k[G]^*$ is a bialgebra.*

Proof: We know that $k[G]^*$ is an algebra and a coalgebra from Examples 1.2.8 and 2.1.4. To prove that $k[G]^*$ is a bialgebra, we will show that condition 2 of Definition 2.1.6 is satisfied.

First we will show that Δ is an algebra map.

$$
\begin{aligned}
1.\quad \text{if } g=h, \text{ then } \Delta(p_g p_h) &= \Delta(p_g)\\
&= \textstyle\sum_{a\in G} p_a\otimes p_{a^{-1}g}\\
&= \textstyle\sum_{a\in G} p_a p_a\otimes p_{a^{-1}g}p_{a^{-1}g}\\
&= (\textstyle\sum_{a\in G} p_a\otimes p_{a^{-1}g})(\textstyle\sum_{a\in G} p_a\otimes p_{a^{-1}g})\\
&= \Delta(p_g)\Delta(p_g)\\
&= \Delta(p_g)\Delta(p_h)
\end{aligned}
$$

$$\begin{aligned}
\text{if } g \neq h, \text{ then } \Delta(p_g)\Delta(p_h) &= (\textstyle\sum_{a\in G} p_a \otimes p_{a^{-1}g})(\sum_{b\in G} p_b \otimes p_{b^{-1}h}) \\
&= \textstyle\sum_{a,b\in G} p_a p_b \otimes p_{a^{-1}g} p_{b^{-1}h} \\
&= \textstyle\sum_{a\in G} p_a \otimes p_{a^{-1}g} p_{a^{-1}h} \\
&= 0 \\
&= \Delta(0) \\
&= \Delta(p_g p_h)
\end{aligned}$$

2. $$\begin{aligned}
\Delta(u_{k[G]^*}(1_k)) &= \Delta(\textstyle\sum_{g\in G} p_g) \\
&= \textstyle\sum_{g\in G} \Delta(p_g) \\
&= \textstyle\sum_{g,h\in G} p_h \otimes p_{h^{-1}g} \\
&= \textstyle\sum_{h\in G} p_h \otimes \sum_{j\in G} p_j \\
&= u_{k[G]^*}(1_k) \otimes u_{k[G]^*}(1_k) \\
&= u_{k[G]^* \otimes k[G]^*}(1_k)
\end{aligned}$$

And now we will show that ϵ is an algebra map. Let $\delta_{g,h} = 0$ if $g \neq h$ and $\delta_{g,h} = 1$ if $g = h$. Then

1. $$\begin{aligned}
\epsilon(p_g p_h) &= \epsilon(\delta_{g,h} p_g) \\
&= \delta_{g,h}\delta_{g,e} \\
&= \delta_{g,e}\delta_{h,e} \\
&= \epsilon(p_g)\epsilon(p_h)
\end{aligned}$$

2. $$\begin{aligned}
\epsilon(u_{k[G]^*}(1_k)) &= \epsilon(\textstyle\sum_{g\in G} p_g) \\
&= \textstyle\sum_{g\in G} \epsilon(p_g) \\
&= 1_k \\
&= u_k(1_k)
\end{aligned}$$

Thus we see that condition 2 holds. Hence $k[G]^*$ is a bialgebra if G is a finite group. □

2.2 Equivalent Conditions for R to be G-Graded

Now we will show a relationship between a G-graded ring, R, and the group ring, $k[G]$. As well, if G is finite, there is also a relationship to the dual of the group ring, $k[G]^*$.

Proposition 2.2.1 *For any group G and ring R the following are equivalent:*

1. *R is a G-graded ring.*

2. *There exists a k-linear map $\psi : R \longrightarrow R \otimes k[G]$ such that for all $r, s \in R$*

   (a) $(I \otimes \epsilon)\psi(r) = r \otimes 1$

   (b) $(I \otimes \Delta)\psi(r) = (\psi \otimes I)\psi(r)$

   (c) $\psi(rs) = \psi(r)\psi(s)$.

   If G is a finite group, then either 1. or 2. is equivalent to

3. *R is a left $k[G]^*$-module such that for all $p_g \in k[G]^*$ and $r, s \in R$, $p_g(rs) = \sum_{h\in G}(p_{gh^{-1}}r)(p_h s)$.*

Proof: (**2** $\Rightarrow$ **1**) Let $\psi : R \longrightarrow R \otimes k[G]$ be such that it satisfies condition 2. Define R_g to be the set of all $r \in R$ such that $\psi(r) = r \otimes g$.

Now take an element of $R_g \cap R_h$ and call it r. Then $\psi(r) = r \otimes g = r \otimes h$. Recall from Example 1.2.5 that $R \otimes k[G]$ is a free R-module with basis $1 \otimes g$, $g \in G$. Thus $r(1 \otimes g) = r(1 \otimes h)$ implies $g = h$, or $r = 0$. Next, take $r \in R$ and let $\psi(r) = \sum_{g\in G} r(g) \otimes g$. By property (b) of condition 2, we have that $\sum_{g\in G} r(g) \otimes g \otimes g = \sum_{g\in G} \psi(r(g)) \otimes g$. This means that $\psi(r(g)) = r(g) \otimes g$, putting $r(g)$ in R_g. Thus we can write r as $\sum_{g\in G} r(g)$.

Let $r \in R_g$ and $s \in R_h$. Then $\psi(rs) = \psi(r)\psi(s) = (r \otimes g)(s \otimes h) = rs \otimes gh$. So $rs \in R_{gh}$. Hence R is G-graded.

(**1** $\Rightarrow$ **2**) Let R be G-graded and define $\psi : R \longrightarrow R \otimes k[G]$ by $\psi(r) = \sum_{g\in G} r_g \otimes g$.

Now $(I \otimes \epsilon)(\sum_{g\in G} r_g \otimes g) = \sum_{g\in G} r_g \otimes \epsilon(g) = \sum_{g\in G} r_g \otimes 1_k = r \otimes 1_k$.

We can see that $(\psi \otimes I)(\psi(r)) = \sum_{g\in G} \psi(r_g) \otimes g = \sum_{g\in G} r_g \otimes g \otimes g$. We can ignore the second sum in this equation, since $(r_g)_h = 0$ when $h \neq g$. Continuing, we observe that this sum is equal to $\sum_{g\in G} r_g \otimes \Delta(g) = (I \otimes \Delta)(\psi(r))$.

Now let $r, s \in R$. Then

$$\begin{aligned}
\psi(rs) &= \textstyle\sum_{g\in G}(rs)_g \otimes g \\
&= \textstyle\sum_{g,h\in G} r_{gh^{-1}} s_h \otimes g \\
&= \textstyle\sum_{x,h\in G} r_x s_h \otimes xh \\
&= \textstyle\sum_{x\in G} r_x \otimes x \sum_{h\in G} s_h \otimes h \\
&= \psi(r)\psi(s).
\end{aligned}$$

For a finite group G we will show that condition 1 is equivalent to condition 3.

(**3** $\Rightarrow$ **1**) Let R be a left $k[G]^*$-module that satisfies condition 3, and let $R_g = \{p_g r | r \in R\}$.

Take $r \in R_g \cap R_h$. Then $r = p_g r = p_h r$. Thus $r = p_h(p_g r) = (p_h p_g)r = 0$ if $g \neq h$. It is clear that each $r \in R$ can be written as a sum of such elements, since $r = (1)(r) = (\sum_{g\in G} p_g)r$.

Next, let $r \in R_g$ and $s \in R_h$. Then $p_{gh}(rs) = \sum_{a \in G}(p_{gha^{-1}}r)(p_a s) = (p_g r)(p_h s) = rs$. So R is a G-graded ring.

(**1** $\Rightarrow$ **3**) Let R be a G-graded ring, and define the action of $k[G]^*$ on R by $p_g r = r_g$. Let us show that R is a left $k[G]^*$-module.

Let $r, s \in R$ and $p_g, p_h \in k[G]^*$. Then $p_g(r+s) = (r+s)_g = r_g + s_g = p_g r + p_g s$. The second condition of Definition 1.1.1 is satisfied by the generalization of the defined action of $k[G]^*$. Finally, $(rp_g)p_h = \delta_{g,h} r_g$, where δ is the Kronecker delta. For the other side of the associativity condition, $r(p_g p_h) = r_g \delta_{g,h}$. Hence R is a $k[G]^*$-module.

Now let $r, s \in R$. Then $(rs)p_g = (rs)_g = \sum_{a \in G} r_a s_{a^{-1}g} = \sum_{a \in G}(rp_a)(sp_{a^{-1}g})$. Therefore condition 3 is satisfied. $\square$

Chapter 3
Smash Products

3.1 The Smash Product of a G-graded Ring

Let G be a finite group, and let R be a G-graded ring. From Proposition 2.2.1, we know that R is a left $k[G]^*$-module. Define $R\#k[G]^*$ to be the k-module, $R \otimes k[G]^*$ with multiplication defined by:

$$(r\#p_g)(s\#p_h) = rs_{gh^{-1}}\#p_h, \tag{3.1}$$

where $r, s \in R$, and p_g, p_h are as defined in Example 1.2.8. This multiplication can be extended to finite sums of such elements. Each element in $R\#k[G]^*$ can be written as a finite sum of elements of the form $r\#p_g$.

We will show that the multiplication described above is associative.

$$\begin{aligned}(r\#p_g)((s\#p_h)(t\#p_m)) &= (r\#p_g)(st_{hm^{-1}}\#p_m) \\ &= r(st_{hm^{-1}})_{gm^{-1}}\#p_m \\ &= rs_{gh^{-1}}t_{hm^{-1}}\#p_m \\ &= (rs_{gh^{-1}}\#p_h)(t\#p_m) \\ &= ((r\#p_g)(s\#p_h))(t\#p_m)\end{aligned}$$

The other ring conditions come from the definition of the tensor product. Hence $R\#k[G]^*$ is a ring.

Proposition 3.1.1 *For a finite group G and a G-graded ring with unity R, $R\#k[G]^*$ is a k-algebra.*

Proof: We have already shown that $R \otimes k[G]^*$ is associative. Define a k-linear map, $u : k \longrightarrow R \otimes k[G]$, by $u(1_k) = 1_R \otimes \sum_{g \in G} p_g$. Then

$$\begin{aligned}M(u \otimes I)(1_k \otimes r\#p_g) &= (1_R\# \textstyle\sum_{h\in G} p_h)(r\#p_g) \\ &= \textstyle\sum_{h \in G} r_{hg^{-1}}\#p_g \\ &= r\#p_g.\end{aligned}$$

The other half of the unitary diagram is satisfied similarly. □

In [18], the smash product of a Hopf algebra, H, and a k-algebra, A, with H-action is defined. The above definition is a special case of this construction, where $A = R$ and $H = k[G]^*$. Another example of the smash product is the skew group ring, defined in Example 1.2.11. In this case, $H = k[G]$, and $A = R$.

Let us now extend the definition of the smash product to rings graded by infinite groups. For this, we will use P_G, the subring of $k[G]^*$ defined in Example 1.2.8.

With both R and P_G being k-modules, we can form the tensor product $R \otimes P_G$. We will use the multiplication defined in Equation 3.1 for the multiplication in this ring. Again, it is extended by linearity to finite sums. This structure gives an associative ring.

Definition 3.1.2 *Let G be any group, and let R be a G-graded ring. Now take P_G to be the k-subalgebra of $k[G]^*$ generated by the projection maps, $\{p_g\}_{g \in G}$. Then the smash product, denoted $R\#P_G$, is the tensor product $R \otimes P_G$ with multiplication as given in Equation 3.1. Elements in $R\#P_G$ are written $\sum_{g \in G} r(g)\#p_g$ with the understanding that only finitely many $r(g)$ are non-zero.*

If G is a finite group, P_G is equal to $k[G]^*$. Thus the smash product above is the same as that defined for finite groups G. For finite G, the identity in $k[G]^*$ is $\sum_{g \in G} p_g$. However, if G is infinite, the smash product $R\#P_G$ has no identity element.

Remark 3.1.3 The group G acts as a group of automorphisms on $R\#P_G$ by $g(r\#p_h) = r\#p_{hg^{-1}}$. Clearly these maps are automorphisms of the additive group $R\#P_G$. To see that $g \in G$ is multiplication perserving, we compute

$$\begin{aligned} g((r\#p_h)(s\#p_t)) &= g(rs_{ht^{-1}}\#p_t) \\ &= rs_{ht^{-1}}\#p_{tg^{-1}} \\ &= (r\#p_{hg^{-1}})(s\#p_{tg^{-1}}) \\ &= g(r\#p_h)g(s\#p_t). \end{aligned}$$

We say that an ideal, I, of $R\#P_G$ is G-invariant if each $g \in G$ maps I to I.

3.2 The Smash Product as a Ring of Matrices

Let G be a group, R a G-graded ring, and let the ring $M_G(R)$ denote the set of row and column finite matrices indexed by G with entries from R. That means that each column and each row has only finitely many non-zero entries. However, the matrices themselves can have an infinite number of entries. We use $M_G^*(R)$ to denote matrices with finitely many entries.

Proposition 3.2.1 *The smash product, $R\#P_G$, can be embedded in the matrix ring $M_G^*(R)$.*

Proof: Let $\phi : R\#P_G \longrightarrow M_G^*(R)$, where $\phi(\sum_{g\in G} r(g)\#p_g) = \sum_{g\in G}\sum_{h\in G} r(g)_h e(hg, g)$. The additive property of homomorphisms is clearly present from the definition. Now let $r\#p_g, s\#p_h \in R\#P_G$. Then

$$\begin{aligned}
\phi((r\#p_g)(s\#p_h)) &= \phi(rs_{gh^{-1}}\#p_h) \\
&= \textstyle\sum_{x\in G}(rs_{gh^{-1}})_x e(xh, h) \\
&= \textstyle\sum_{x\in G} r_{xhg^{-1}} s_{gh^{-1}} e(xh, h) \\
&= \textstyle\sum_{y\in G} r_y s_{gh^{-1}} e(yg, h) \\
&= \textstyle\sum_{y\in G} r_y e(yg, g) \sum_{z\in G} s_z e(zh, h) \\
&= \phi(r\#p_g)\phi(s\#p_h).
\end{aligned}$$

Now to show that ϕ is injective, let $\phi(\sum_{g\in G} r(g)\#p_g) = \phi(\sum_{g\in G} s(g)\#p_g)$. Then we have

$$\begin{aligned}
\textstyle\sum_{g,h\in G} r(g)_h e(hg, g) &= \textstyle\sum_{g,h\in G} s(g)_h e(hg, g) \\
r(g)_h &= s(g)_h \ (\forall g, h \in G) \\
r(g) &= s(g) \\
\textstyle\sum_{g\in G} r(g)\#p_g &= \textstyle\sum_{g\in G} s(g)\#p_g.
\end{aligned}$$

Hence ϕ is an injective homomorphism. Therefore we can consider the smash product as a ring of matrices. □

3.3 Adjoining a 1 to the Smash Product

Since we would like to consider R as a subring of the smash product, $R\#P_G$, in the infinite case, we need to find a way to adjoin a 1 to the smash product. One method of adjoining the 1 is called the Dorroh extension. Let us describe the method for a general ring.

Let R be a ring without a 1. The Dorroh extension is a ring with 1 that contains R as a subring. It involves forming $\mathbf{Z} \times R$ and defining a ring structure. First let the addition and multiplication on $\mathbf{Z} \times R$ be defined by

$$(n, r) + (m, s) = (n + m, r + s) \tag{3.2}$$

and

$$(n, r)(m, s) = (nm, ns + rm + rs) \tag{3.3}$$

Definition 3.3.1 *For a ring R without unity, the ring described above is called the Dorroh extension of R. We write it as $(\mathbf{Z}, R)$ or sometimes R^1.*

Example 3.3.2 In the definition of the smash product of a G-graded ring, R, where G is an infinite group, $R\#P_G$ is a ring without a 1. We will construct an object similar to that of the Dorroh extension, using R as the ring with unity instead of $\mathbf{Z}$. Let R act upon $R\#P_G$ by

$$r(s\#p_g) = rs\#p_g \tag{3.4}$$

and

$$(s\#p_g)r = \sum_{x\in G} sr_{gx^{-1}}\#p_x. \tag{3.5}$$

These actions make the smash product a left and right R-module. We will show that it also satisfies the bimodule condition in Definition 1.1.3. Let $r, s \in R$ and $t\#p_g \in R\#P_G$. Then

$$\begin{aligned}(rt\#p_g)s &= \textstyle\sum_{x\in G} rts_{gx^{-1}}\#p_x\\ &= \textstyle\sum_{x\in G} rts_{gx^{-1}}\#p_x\\ &= r\textstyle\sum_{x\in G} ts_{gx^{-1}}\#p_x\\ &= r(t\#p_g s).\end{aligned}$$

These actions make $R\#P_G$ an R-bimodule.

So we can extend the smash product to $(R, R\#P_G)$ with multiplication defined as in the case of the Dorroh extension. †

Proposition 3.3.3 *For a G-graded ring R with unity, $(R, R\#P_G)$ is an associative ring with unity, which contains $R\#P_G$ as a subring.*

Proof: Clearly, $(R, R\#P_G)$ is an abelian group since both R and $R\#P_G$ are rings. So, we just have to check distributivity and associativity.

Let $(r, a\#p_g), (s, b\#p_h), (t, c\#p_m) \in (R, R\#P_G)$, and we will let b and c be homogeneous elements from $R_{gh^{-1}}$ and $R_{hm^{-1}}$ respectively. Then

1. $(r, a\#p_g)((s, b\#p_h) + (t, c\#p_m))$

$$\begin{aligned} &= (r, a\#p_g)(s + t, b\#p_h + c\#p_m)\\ &= (r(s + t), r(b\#p_h + c\#p_m) + a\#p_g(b\#p_h + c\#p_m) + a\#p_g(s + t))\\ &= (rs + rt, rb\#p_h + rc\#p_m + a\#p_g b\#p_h + a\#p_g c\#p_m + a\#p_g s + a\#p_g t)\\ &= (rs, rb\#p_h + a\#p_g b\#p_h + a\#p_g s) + (rt, rc\#p_m + a\#p_g c\#p_m + a\#p_g t)\\ &= (r, a\#p_g)(s, b\#p_h) + (r, a\#p_g)(t, c\#p_m)\end{aligned}$$

2. $((r, a\#p_g) + (s, b\#p_h))(t, c\#p_m)$

$$
\begin{aligned}
&= (r+s, a\#p_g + b\#p_h)(t, c\#p_m) \\
&= ((r+s)t, (r+s)c\#p_m + (a\#p_g + b\#p_h)c\#p_m + (a\#p_g + b\#p_h)t) \\
&= (rt + st, rc\#p_m + sc\#p_m + a\#p_g c\#p_m + b\#p_h c\#p_m + a\#p_g t + b\#p_h t) \\
&= (rt, rc\#p_m + a\#p_h c\#p_m + a\#p_g t) + (st, sc\#p_m + a\#p_g c\#p_m + b\#p_h t) \\
&= (r, a\#p_g)(t, c\#p_m) + (s, b\#p_h)(t, c\#p_m)
\end{aligned}
$$

3. $(r, a\#p_g)((s, b\#p_h)(t, c\#p_m))$

$$
\begin{aligned}
&= (r, a\#p_g)(st, sc\#p_m + bc\#p_m + \textstyle\sum_{x\in G} bt_{hx^{-1}}\#p_x) \\
&= (r(st), r(sc\#p_m + bc\#p_m + \textstyle\sum_{x\in G} bt_{hx^{-1}}\#p_x) \\
&\quad + a\#p_g(sc\#p_m + bc\#p_m + \textstyle\sum_{x\in G} bt_{hx^{-1}}\#p_x) + a\#p_g(st)) \\
&= ((rs)t, r(sc)\#p_m + r(bc)\#p_m + \textstyle\sum_{x\in G} r(bt_{hx^{-1}})\#p_x \\
&\quad + a(sc)_{gm^{-1}}\#p_m + a(bc)\#p_m + \textstyle\sum_{x\in G} a(bt_{hx^{-1}})_{gx^{-1}}\#p_x \\
&\quad + \textstyle\sum_{y\in G} a(st)_{gy^{-1}}\#p_y) \\
&= ((rs)t, (rs)c\#p_m + (rb\#p_h)c\#p_m + (rb\#p_h)t \\
&\quad + a(s_{gh^{-1}}c)\#p_m + a(bc)\#p_m + \textstyle\sum_{x\in G} a(bt_{hx^{-1}})\#p_x \\
&\quad + \textstyle\sum_{y,v\in G} a(s_v t_{v^{-1}gy^{-1}})\#p_y) \\
&= ((rs)t, (rs)c\#p_m + (rb\#p_h)c\#p_m + (rb\#p_h)t \\
&\quad + (as_{gh^{-1}}c\#p_m + (a\#p_g b\#p_h)c\#p_m + (a\#p_g b\#p_h)t \\
&\quad + (\textstyle\sum_{v\in G} as_v\#p_{v^{-1}g})t) \\
&= ((rs)t, (rs)c\#p_m + (rb\#p_h)c\#p_m + (rb\#p_h)t \\
&\quad + (\textstyle\sum_{x\in G} as_{gx^{-1}}\#p_x)c\#p_m + (a\#p_g b\#p_h)c\#p_m + (a\#p_g b\#p_h)t \\
&\quad + (\textstyle\sum_{y\in G} as_{gy^{-1}}\#p_y)t) \\
&= ((rs)t, (rs)c\#p_m + (rb\#p_h)c\#p_m + (rb\#p_h)t \\
&\quad + (a\#p_g s)c\#p_m + (a\#p_g b\#p_h)c\#p_m + (a\#p_g b\#p_h)t \\
&\quad + (a\#p_g s)t) \\
&= ((rs)t, (rb\#p_h + a\#p_g b\#p_h + a\#p_g s)t \\
&\quad + (rb\#p_h + a\#p_g b\#p_h a\#p_g s)c\#p_m + (rs)c\#p_m) \\
&= (rs, rb\#p_h + a\#p_g b\#p_h + a\#p_g s)(t, c\#p_m) \\
&= ((r, a\#p_g)(s, b\#p_h))(t, c\#p_m).
\end{aligned}
$$

The 1 of this ring is $(1_R, 0)$. Also, $R\#P_G$ is isomorphic to the subring $(0, R\#P_G)$. □

Lemma 3.3.4 *Let R be a G-graded ring with unity. Then*

1. *$R\#P_G \cong (0, R\#P_G)$ is an ideal of $(R, R\#P_G)$*
2. *$(R, R\#P_G)/(0, R\#P_G) \cong R$.*

Proof:

1. Let $(0, r) \in (0, R\#P_G)$ and $(s, t) \in (R, R\#P_G)$. Then $(0, r)(s, t) = (0, rs+rt) \in (0, R\#P_G)$. Similarly, $(s, t)(0, r) = (0, sr + tr) \in (0, R\#P_G)$.

2. Let $\phi : (R, R\#P_G) \longrightarrow R$ be given by $\phi((s, r)) = s$. First, let us show that ϕ is a ring homomorphism. Clearly the additive property holds, since addition is performed component-wise in $(R, R\#P_G)$. As for the multiplicative property, the multiplication in $(R, R\#P_G)$ is component-wise in the first component.

   Next, we will note that the kernel of ϕ is $(0, R\#P_G)$, and that ϕ is onto. By the first isomorphism theorem (see [15, Theorem 3.4]) which states that for a homomorphism $\tau : A \longrightarrow B$, $A/ker(\tau) \cong im(\tau)$, we have that $(R, R\#P_G)/(0, R\#P_G) \cong R$.

□

In [13], Quinn defines a smash product for G-graded rings with unity with G being an infinite group. There, the definition relies on the ring of matrices, $M_G(R)$, mentioned in Section 3.2. First, the ring R is embedded in the matrix ring by the homomorphism $\eta : R \longrightarrow M_G(R)$ where $\eta(r) = A$ with $A(g, h) = r_{gh^{-1}}$. The image of η is denoted $\overline{R}$ and for $r \in R$, we write $\overline{r}$ for $\eta(r)$.

Now take the subring of $M_G(R)$ generated by $\overline{R}$ and the matrices $\{e(g, g)\}_{g\in G}$. We will call this ring $M_{G(R)}$.

Theorem 3.3.5 *For a G-graded ring R, $M_{G(R)}$ is isomorphic to $(R, R\#P_G)$.*

Proof: Let $\phi : M_{G(R)} \longrightarrow (R, R\#P_G)$ be defined by $\phi(\overline{r} + \overline{s}e(g, g)) = (r, s\#p_g)$, extended to sums by linearity. This satisfies the additive property of homomorphisms. For the multiplicative property, we will need the following:

$$\begin{aligned}
\overline{r}e(g, g)\overline{s} &= \textstyle\sum_{a,b\in G} r_{ab^{-1}} e(a, b)e(g, g) \sum_{c,d\in G} s_{cd^{-1}} e(c, d) \\
&= \textstyle\sum_{a,b,c,d\in G} r_{ab^{-1}} s_{cd^{-1}} e(a, b)e(g, g)e(c, d) \\
&= \textstyle\sum_{a,b,d\in G} r_{ab^{-1}} s_{gd^{-1}} e(a, b)e(g, d) \\
&= \textstyle\sum_{a,d\in G} r_{ag^{-1}} s_{gd^{-1}} e(a, d) \\
&= \textstyle\sum_{a,d\in G} (r s_{gd^{-1}})_{ad^{-1}} e(a, d)e(d, d) \\
&= \textstyle\sum_{a,b,d\in G} (r s_{gd^{-1}})_{ab^{-1}} e(a, b)e(d, d) \\
&= \textstyle\sum_{d\in G} \overline{r s_{gd^{-1}}} e(d, d).
\end{aligned}$$

Now take $\overline{r}+\overline{s}e(g,g), \overline{t}+\overline{u}e(h,h) \in M_{G(R)}$. Then

$$\begin{aligned}\phi((\overline{r}+\overline{s}e(g,g))(\overline{t}+\overline{u}e(h,h))) &= \phi(\overline{rt}+\overline{ru}e(h,h)+\textstyle\sum_{x\in G}\overline{st_{gx^{-1}}}e(x,x)+\overline{su_{gh^{-1}}}e(h,h)) \\ &= (rt, ru\#p_h+\textstyle\sum_{x\in G} st_{gx^{-1}}\#p_x + su_{gh^{-1}}\#p_h) \\ &= (rt, r(u\#p_h)+(s\#p_g)t+(s\#p_g)(u\#p_h)) \\ &= (r, s\#p_g)(t, u\#p_h) \\ &= \phi(\overline{r}+\overline{s}e(g,g))\phi(\overline{t}+\overline{u}e(h,h).\end{aligned}$$

Therefore ϕ is a homomorphism. The kernel of ϕ is 0 making the homomorphism injective. That it is surjective is clear, since $(r, s\#p_g) = \phi(\overline{r}+\overline{s}e(g,g))$. Hence ϕ is a ring isomorphism. □

With this equivalence shown, we will freely use either notation in computation.

Corollary 3.3.6 *For infinite groups, $R\#P_G$ is isomorphic to $\bigoplus_{g\in G}\overline{R}e(g,g)$.*

Proof: Using the isomorphism from Theorem 3.3.5, the equivalence is clear, since $\phi(\overline{r}e(g,g)) = r\#p_g$ for all $r \in R$ and all $g \in G$. □

The smash product, $R\#k[G]^*$, has been used to study properties of a ring R graded by a finite group. For example, Cohen and Montgomery proved in [5, Proposition 4.3] that for the Jacobson radical, J, and a G-graded ring, R, the graded Jacobson radical of R was equal to the largest graded ideal contained in the Jacobson radical of R; that is, $J_G(R) = J(R)_G$. Another result from the same paper is that $J(R_e) = J_G(R) \cap R_e$, for a G-graded ring [5, Corollary 4.2]. The same result was also proven in [5, Corollary 5.4] for the prime radical.

In [13], Quinn uses the ring $M_{G(R)}$ to examine properties of graded rings. And Saorin, in [17], computes the Jacobson radical of $M_{G(R)}$. In Chapter 5, we will extend that result to all hereditary radicals. However, we will first discuss graded radicals in Chapter 4.

Chapter 4

Radicals of the Smash Product

From now on, we will assume that k is the ring of integers.

4.1 Definition of Radical Classes of Rings

Definition 4.1.1 *Let λ be a class of associative rings. We say that λ is a radical class if*

1. *λ is homomorphically closed;*
2. *if A/B and B are in λ, then A is in λ;*
3. *if I_α, $\alpha \in \Delta$, where Δ is any indexing set, is an ascending chain of ideals of A with each I_α in λ, then $\bigcup_\alpha I_\alpha$ is in λ.*

We denote by $\lambda(R)$ the largest ideal of R that is in λ. We call $\lambda(R)$ the λ radical of R. For example, if λ is the Jacobson radical class, we would say that $\lambda(R)$ was the Jacobson radical of R. This characterization can be found in [6, §1.1].

Note that if α is an automorphism of the ring R, and I is an ideal of R in λ, then $\alpha(I) \in \lambda$ also by Definition 4.1.1 (1). Since $\lambda(R)$ is the largest ideal of R in λ, $\alpha(\lambda(R)) = \lambda(R)$.

Example 4.1.2 An example of a radical class is $D = \{R \mid \forall n \in \mathbf{Z}\ \ \forall r \in R\ \ \exists y \in R$ such that $r = ny\}$. This radical is called the divisible radical.

It is clear that D is homomorphically closed since, taking any homomorphism ϕ, $\phi(x) = \phi(ny) = n\phi(y)$. For the second condition, we see that for any $a \in A$, $a \in \alpha + B \in A/B$ for some $\alpha \in A$. Now $A/B \in D$ so $\alpha + B = n(\alpha_1 + B) = n\alpha_1 + B$. So we now have that $a = n\alpha_1 + c$ for some $c \in B$. Since $c \in B \in D$, we get that $a = n\alpha_1 + nc_1 = n(\alpha_1 + c_1)$. Therefore $A \in D$. The final condition is easily shown.

For $i = \sum_{\alpha \in \Delta} i_\alpha$ where each $i_\alpha \in I_\alpha$, $i = \sum_{\alpha \in \Delta} nj_\alpha = n \sum_{\alpha \in \Delta} j_\alpha$, where $i_\alpha = nj_\alpha$, $j_\alpha \in I_\alpha$.

Therefore, D is indeed a radical class. †

Radicals which depend only on the additive structure of the ring are called A-radicals [9]; the radical D above is an example of an A-radical.

Before going further, let us define R^+ to the additive group of the ring R. Now we can say when a radical is an A-radical.

Definition 4.1.3 *A radical class, $\mathcal{A}$, of rings is called an A-radical class if it satisfies the following condition:*

$$R \in \mathcal{A}, \;\; R^+ \cong S^+ \Rightarrow S \in \mathcal{A}.$$

[9, Definition 1.2]

Since A-radicals depend only on the additive group, we have that $\lambda(R \bigoplus R) = \lambda(R) \bigoplus \lambda(R)$ for an A-radical λ. This can be extended to an arbitrary number of direct sums. With this piece of information, we note that the additive group of $(R, R\#P_G)$ is isomorphic to $R^+ \bigoplus (R\#P_G)^+$, and $(R\#P_G)^+ = \bigoplus_{g \in G} R^+$. Thus, for an A-radical, λ, we have that $\lambda((R, R\#P_G)) = \lambda(R) \bigoplus (\bigoplus_{g \in G} \lambda(R))$.

Now let us move on to the Jacobson and prime radicals. In [6], equivalent definitions are given for the Jacobson radical. But before we give them, let us define a few terms.

Definition 4.1.4 *We say that a right ideal, I, is regular if there is an element $x \in R$ such that $xr - r \in I$ for all $r \in R$. Left regularity is defined similarly.*

Definition 4.1.5 *We say that an element $r \in R$ is right quasi-regular if there is an element $x \in R$ such that $r + x + rx = 0$. Left quasi-regularity is defined in the same manner.*

Definition 4.1.6 *For a ring R, we define the Jacobson radical by any one of the following characterizations. The Jacobson radical of R, $J(R)$ is*

1. *the intersection of all regular maximal left (right) ideals of R [6]*
2. *the set of all $r \in R$ such that xr is left (right) quasi-regular for every $x \in R$ [6, 12]*

3. *the set of all $r \in R$ such that $rM = 0$ for any left R-module M such that M has no proper nontrivial R-submodules. [14, 10].*

Note that if R does have unity, then every right (left) ideal is regular, with the x in this case being 1, and that an element r, being right (left) quasi-regular is the same as $(1+r)$ being right (left) invertible, meaning that the element has an inverse that is multiplied on the right (left). So this allows simpler characterizations for the Jacobson radical when R is a ring with unity.

Example 4.1.7 Let $\mathbf{Q_0}$ be the set of rational numbers with the usual addition. Now define the multiplication as follows: $q_1 q_2 = 0$ for any $q_1, q_2 \in \mathbf{Q_0}$. With these operations, $\mathbf{Q_0}$ is a commutative ring without unity. What is the Jacobson radical of this ring? Since $\mathbf{Q_0}$ does not have a 1, we use Definition 4.1.6 (2). That means we need to find elements $q \in \mathbf{Q_0}$ such that for every $x \in \mathbf{Q_0}$ we can find a $y \in \mathbf{Q_0}$ such that $xq + y + yxq = 0$. However, since the multiplication yields 0 in all cases, the condition simplifies to $y = 0$. Thus, every element of $\mathbf{Q_0}$ is left quasi-regular. Therefore $J(\mathbf{Q_0}) = \mathbf{Q_0}$. †

Definition 4.1.8 *An element $r \in R$ is strongly nilpotent if for every sequence $x_0, x_1, \cdots$, where $x_0 = r$ and $x_{n+1} \in x_n R x_n$, there exists an $N \in \mathbf{Z}$ such that $x_n = 0$ for all $n \geq N$. Every strongly nilpotent element, r, is also nilpotent meaning $r^n = 0$ for some $n > 0$ [11, §3.2].*

Definition 4.1.9 *Recall from Definition 1.3.5 the definition of a prime ideal. The prime radical, denoted $N(R)$, is the intersection of all prime ideals of the ring R [6, Theorem 18]. It is also the set of all strongly nilpotent elements [11, §3.2, Proposition 1]. If R is a commutative ring, $N(R)$ is the set of all nilpotent elements [10].*

Example 4.1.10 Let $\mathbf{Q}[x]$ be the polynomial ring over the rational numbers. It is clear that this ring is commutative and has a 1. Since $\mathbf{Q}$ is a field, it has no nilpotent elements. It follows that the polynomial ring has no nilpotent elements. Therefore $N(\mathbf{Q}[x]) = 0$. †

Definition 4.1.11 *We call a radical, λ, hereditary if, for any ideal I of R, $\lambda(I) = \lambda(R) \cap I$ [6].*

It is relevant to note that both the Jacobson and prime radicals are hereditary [6]. An equivalent way of defining a hereditary radical is as follows: if λ is a radical class, then λ is hereditary if when $R \in \lambda$ and S is an ideal of R, then $S \in \lambda$.

Using this definition, we can see that the divisible radical, D, from Example 4.1.2, is not hereditary. For the ring $\mathbf{Q_0}$ from Example 4.1.7 lies in D because for all $x \in \mathbf{Q_0}$, all positive integers n, $x = \frac{x}{n} + \cdots + \frac{x}{n}$ (n factors) $= n(\frac{x}{n})$. However the subset $\mathbf{Z}$ of $\mathbf{Q}$ is an ideal of $\mathbf{Q_0}$ and is not in D.

4.2 Graded Radicals

A graded radical is a radical in the class of graded rings. From now on, R will denote a G-graded ring with 1, and G will be a fixed group. Let us take a look at the traditional graded Jacobson radical. We will list the equivalent definitions for the graded Jacobson radical, $J_G(R)$.

Definition 4.2.1 *For a G-graded ring R with 1, $J_G(R)$ is the graded ideal defined by any of the following equivalent statements:*

1. *the set of elements of R annihilating all G-graded R-modules that have no proper nontrivial graded R-submodules. We call such modules simple G-graded R-modules [4, 3].*
2. *the intersection of all left (right) ideals of R maximal in the set of graded left (right) ideals of R [3].*

Theorem 4.2.2 *If R is a G-graded ring then $J(R\#P_G) = J_G(R)\#P_G$ [3, Theorem 3.2].*

Proof: As given in [3].

First, we show that $J(R\#P_G)$ is included in $J_G(R)\#P_G$. Suppose $x = \sum_{g\in G} r(g)\#p_g \in J(R\#P_G)$. Since $J(R\#P_G)$ is a two-sided ideal, $r(g)\#p_g \in J(R\#P_G)$. Therefore it suffices to show that $r\#p_g \in J(R\#P_G)$ implies r is in $J_G(R)$. Note that since $J(R\#P_G)$ is G-invariant under the G-action on $R\#P_G$ defined in Remark 3.1.3, $r\#p_g \in J(R\#P_G)$ implies $r\#p_h \in J(R\#P_G)$ for all h in G. To show that r is in $J_G(R)$, we show that r annihilates all simple graded left R-modules.

Let N be a simple graded left R-module. Then N is a unital left $R\#P_G$-module with module action given by $r\#p_g n = rn_g$, and is simple by [3, Corollary 2.5] and [3, Theorem 2.6]. Therefore for any $n \in N$ and for all $g \in G$,

$$rn_g = (r\#p_g)n = 0.$$

Therefore r annihilates N and r is in $J_G(R)$.

Conversely, let $x = \sum_{g\in G} r(g)\#p_g$ be an element of $J_G(R)\#P_G$. Since $J_G(R)$ is G-graded, we may assume that each $r(g)$ is homogeneous. Let M be a simple left $R\#P_G$-module. Then M is a simple graded left R-module where $M_g = p_g(M)$. The module action is given by $rm_g = r\#p_g m$, and so $r\#p_g m_h = 0$ for $g \neq h$. So M is annihilated by each $r(g)$. Thus, for each $g \in G$,

$$r(g)\#p_g M = r(g)M_g = 0,$$

and x annihilates M. □

In Definition 4.2.1 part (2), we define the graded Jacobson radical using graded maximal left (right) ideals. We can define the graded prime radical using graded prime ideals. We say a graded ideal, I, is graded prime, if, for graded ideals A and B, $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$.

Definition 4.2.3 *For a G-graded ring R, the graded prime radical $N_G(R)$ is the intersection of all graded prime ideals of R. Also, by [5, Theorem 5.1], we have that $N_G(R) = N(R)_G$; that is, the largest graded ideal of R contained in $N(R)$.*

The reflected radical is a radical in the class of graded rings. The definition of this radical depends on the smash product.

Definition 4.2.4 *Let λ be a radical class, and let the reflected radical of the graded ring R, $\lambda_{ref}(R)$ be the set $\{r \in R | r\#p_g \in \lambda(R\#P_G)\ \forall g \in G\}$ [2]. The radical class for the reflected radical of λ is given as $\lambda_{ref} = \{R | R$ is a G-graded ring with $R\#P_G \in \lambda\}$ [1, Proposition 1.2].*

Proposition 4.2.5 *For any radical, λ, $\lambda(R\#P_G) = \lambda_{ref}(R)\#P_G$, for any G-graded ring R [1].*

However, before we prove this, we must define a few terms. First, for any ideal I of $R\#P_G$, we say that $I_R = \{r | r\#p_g \in I,\ \forall g \in G\}$. Next we set $I^{\downarrow} = (I_R)_G$, the largest graded ideal contained in I_R. It can be seen that if R is a ring with 1, then I_R is always graded. If we take $x \in I_R$ then $x\#p_g \in I$ for all $g \in G$. Then we get that $(1\#p_{hg})(x\#p_g) = x_h\#p_g \in I$ for all $g, h \in G$. Hence $x_h \in I_R$ for all $h \in G$. Therefore I_R is graded.

Lemma 4.2.6 *Let I be an ideal of $R\#P_G$, where R is a ring with 1. If I is G-invariant, then $I = I^{\downarrow}\#P_G$.*

Proof: Let $x = \sum_{g\in G} x(g)\#p_g$ be in I. Then $x(1\#p_h) = x(h)\#p_h$. So the elements $x(g)$ are in I, for any $x \in I$. Let us concentrate on $x(g)\#p_g$. Since I is G-invariant, $h(x(g)\#p_g) = x(g)\#p_{gh^{-1}} \in I$ for all $h \in G$. Hence $x(g) \in I_R$. Now remember that I_R is graded, so $I^{\downarrow} = I_R$. So it is clear that $I = I^{\downarrow}\#P_G$. □

Since we are only considering rings with 1, then we know from above that $\lambda(R\#P_G)_R$ is a graded ideal of R. Hence from Lemma 4.2.6, $\lambda(R\#P_G) = (\lambda(R\#P_G))^{\downarrow}\#P_G$. This means that $(\lambda(R\#P_G))^{\downarrow}$ is the largest ideal, K, of R such that the smash product, $K\#P_G$, is in λ. This is exactly the definition of $\lambda_{ref}(R)$. So $\lambda(R\#P_G) = \lambda_{ref}(R)\#P_G$.

With Theorem 4.2.2, and Proposition 4.2.5, it is plain to see that for the Jacobson radical, $J_G(R) = J_{ref}(R)$ for any G-graded ring R.

In Definition 4.1.9 we gave an element-wise definition of the prime radical. Similarly, we can also describe the reflected prime radical by its elements. To do so, we need to define a new term.

Definition 4.2.7 *Let R be a G-graded ring and let $x = \Sigma_{i=1}^n x_{g_i}$ be an element of R. Then we say that x is graded strongly nilpotent, if for each x_{g_i}, every sequence $x_0, x_1, \cdots$, where $x_0 = x_{g_i}$ and $x_{n+1} \in x_n R_{g_i^{-1}} x_n$, there exists an $N \in \mathbf{Z}$ such that $x_n = 0$ for all $n \geq N$.*

Now, we give a new characterization of $N_{ref}(R)$.

Theorem 4.2.8 *Let $x = \sum_{i=1}^n x_{g_i}$ be in a G-graded ring R. Then x is in $N_{ref}(R)$ if and only if x is graded strongly nilpotent.*

Proof: Take $x = \sum_{g\in G} x_g \in N_{ref}(R)$. Let $h \in G$ and let $z_0 = x_h$. Let a sequence $z_1, z_2 \cdots$ be chosen such that $z_{n+1} \in z_n R_{h^{-1}} z_n$. Since $z_0 \in R_h$, all z_n will be in R_h for $n \geq 0$. So $z_{n+1} = z_n r_n z_n$ where r_n is an element of $R_{h^{-1}}$.

Let us form a new sequence $\{y_n\}$, where $y_n = z_n\#p_e$. Then

$$y_{n+1} = (z_n r_n z_n\#p_e) = (z_n\#p_e)(r_n\#p_h)(z_n\#p_e) \in z_n\#p_e(R\#P_G)z_n\#p_e$$

Since $N_{ref}(R)$ is a graded ideal that contains x, it also contains z_0. Thus, $y_0 \in N_{ref}(R)\#P_G = N(R\#P_G)$. So y_0 is strongly nilpotent, implying that $y_n = 0$ for all n greater than some integer M. Hence $z_n = 0$ for all $n \geq M$.

Because the h was chosen arbitrarily, x is graded strongly nilpotent.

For the converse, take $x \notin N_{ref}(R)$. This implies that $x_g \notin N_{ref}(R)$ for some $g \in G$. Let $y_0 = x_g\#p_e$ which is not in $N(R\#P_G)$, and let $z_0 = x_g$. Since $y_0 \notin N(R\#P_G)$, it is not strongly nilpotent. That means that there is a sequence $y_1, y_2, \cdots$ such that $y_{n+1} \in y_n(R\#P_G)y_n$, which is never zero. Note that $y_1 \in x_g R_{g^{-1}} x_g\#p_e$ and let $y_1 = z_1\#p_e$. Similarly, $y_2 \in z_1 R_{g^{-1}} z_1\#p_e$, etc. Let z_n be such that $y_n = z_n\#p_e$. Since $x_g \in R_g$, all z_n will also be in R_g, with $z_{n+1} \in z_n R_{g^{-1}} z_n$. Since $y_n \neq 0$ for all n, $z_n \neq 0$. Hence x is not graded strongly nilpotent. □

From this characterization, we obtain a new proof of [2, Proposition 2.1] for the prime radical.

Corollary 4.2.9 *If R is a G-graded ring, then $N_{ref}(R) \cap R_e = N(R_e)$.*

Proof: If $x \in N_{ref}(R) \cap R_e$, then $x \in R_e$, which is a subring of R. Also, x is graded strongly nilpotent, since it is in $N_{ref}(R)$. That means that every sequence of the form $x_0, x_1 \cdots$ where $x_0 = x$ and $x_{n+1} \in x_n R_e x_n$ is zero after a finite number of terms. So $N_{ref}(R) \cap R_e \subseteq N(R_e)$.

Conversely, for $x \in N(R_e)$, $x \in R_e$ and is strongly nilpotent. But it is also graded strongly nilpotent, so is in $N_{ref}(R)$. □

With this characterization, we can show the relationship between the graded prime radical $N_G(R)$ and the reflected prime radical, $N_{ref}(R)$. The relationship is already given in [1, Theorem 2.3], but this characterization can give a new proof to a known result.

Corollary 4.2.10 *For any G-graded ring R, $N(R)_G = N_G(R) \subseteq N_{ref}(R)$.*

Proof: Since $N(R)_G$ is graded, we need only consider homogeneous elements. Let $x_g \in N(R)_G \subseteq N(R)$. Then x_g is strongly nilpotent, meaning that all sequences of the form $x_0 = x_g$ and $x_{n+1} \in x_n R x_n$ are zero after a finite number of terms. That implies that all sequences of the form $x_0 = x_g$ and $x_{n+1} \in x_n R_{g^{-1}} x_n$ are zero after a finite number of terms. So $x_g \in N_{ref}(R)$ for all $g \in G$. □

Example 4.2.11 Let $R = \mathbf{Q}[x]$ as in Example 1.2.4. R is a $\mathbf{Z}$-graded ring with $R_n = \{0\}$ if $n < 0$ and $R_n = \mathbf{Q}x^n$ if $n \geq 0$. By Theorem 4.2.8, we have a characterization of the reflected prime radical. Let $f(x) \in R$. If $f(x)$ has no constant term then $f(x)$ is graded strongly nilpotent. This can be easily seen by the grading of R, since for $n > 0$, $R_{n^{-1}} = R_{-n} = \{0\}$. So $xR \subseteq N_{ref}(R)$. From Example 4.1.10 we know that $N(R) = 0$. So we have an example where $N_G(R) \subset N_{ref}(R)$, since $N_G(R) \subseteq N(R) = 0$. †

Chapter 5

Hereditary Radicals of the Smash Product with 1 Adjoined

5.1 Preliminaries

The following theorem from [7], together with the Proposition 4.2.5, leads us to ask if we can characterize the radical of the smash product with a 1 adjoined, for any radical.

Theorem 5.1.1 *Let λ be a ring radical, and let R be a ring. The following are equivalent:*

1. *There exists an extension $(\lambda(R))^*$ of $\lambda(R)$ by R such that $\lambda((\lambda(R))^*) = \lambda(R)$.*
2. $\lambda(R) = 0$.
3. *For every extension A^* of a ring A by R, $\lambda(A) = \lambda(A^*)$.*

[7]

Example 5.1.2 Let R be the polynomial ring $\mathbf{Q}[x]$. Since R is a $\mathbf{Z}$-graded ring, we can form $R\#P_{\mathbf{Z}}$. Also, from Example 4.1.10 we see that $N(R) = 0$. Adjoining a 1 to the smash product, we get the extension $(R, R\#P_{\mathbf{Z}})$. From Theorem 5.1.1, $N((R, R\#P_{\mathbf{Z}})) = (0, N(R\#P_G)) \cong N(R\#P_{\mathbf{Z}}) = N_{ref}(R)\#P_{\mathbf{Z}}$. †

Theorem 4.2.2 leads to the question of what the Jacobson radical of $(R, R\#P_G)$ looks like. And the above proposition suggests it could just be the Jacobson radical of $R\#P_G$. However, from Saorin [17], this is not so. The problem is that $J(R)$ may be nonzero.

Theorem 5.1.3 *If R is a ring with unity graded by an infinite group G, then*

$$J((R, R\#P_G)) = (J_G(R) \cap J(R), J_G(R)\#P_G).$$

[17, Theorem 1]

In the proof, Saorin used Definition 4.1.6 (3) to define the Jacobson radical. We use a different approach to extend his result to all hereditary radicals λ.

5.2 Hereditary Radicals of $(R, R\#P_G)$

Lemma 5.2.1 *If I, J are ideals of R, with J being a graded ideal, such that $I \subseteq J$, then $(I, J\#P_G) \lhd (R, R\#P_G)$.*

Proof: We need only show that the inclusion holds for single components of the ideal and the ring. The general case can easily be deduced from that. Let $(r, s\#p_g) \in (R, R\#P_G)$ and $(i, j\#p_h) \in (I, J\#P_G)$. Then $(r, s\#p_g)(i, j\#p_h) = (ri, r(j\#p_h) + (s\#p_g)i + sj_{gh^{-1}}\#p_h)$. Since J is graded and $I \subseteq J$, the homogeneous components of i and j are in J. Thus, with the help of Equations 3.4 and 3.5, we see that the $R\#P_G$ component of the product is indeed in $J\#P_G$. The right hand side is shown similarly. □

Lemma 5.2.2 *Let R be a G-graded ring. Then for any radical, λ, $\lambda((R, R\#P_G)) \subseteq (\lambda(R), R\#P_G)$.*

Proof: From Lemma 5.2.1 we can see that both $(\lambda(R), R\#P_G)$ and $(0, R\#P_G)$ are ideals of $(R, R\#P_G)$. To show the containment, from [16, Remark 2.6.0], all we need to show is that

$$\lambda\left(\frac{(R, R\#P_G)}{(\lambda(R), R\#P_G)}\right) = 0. \tag{5.1}$$

To accomplish this, we will show that the quotient ring is isomorphic to $R/\lambda(R)$.

By Theorem 3.3.4 (2), $(R, R\#P_G)/(0, R\#P_G) \cong R$ and by a similar proof we get $(\lambda(R), R\#P_G)/(0, R\#P_G) \cong \lambda(R)$. So

$$\frac{R}{\lambda(R)} \cong \frac{\frac{(R,R\#P_G)}{(0,R\#P_G)}}{\frac{(\lambda(R),R\#P_G)}{(0,R\#P_G)}} \cong \frac{(R, R\#P_G)}{(\lambda(R), R\#P_G)}.$$

And since $\lambda(R/\lambda(R)) = 0$, we know that Equation 5.1 is true. □

Since $(0, R\#P_G)$ is an ideal of $(R, R\#P_G)$ we get that for any hereditary radical, λ, $\lambda((0, R\#P_G)) = \lambda((R, R\#P_G)) \cap (0, R\#P_G)$. Now we come to the central theorem, characterizing the radical of the smash product with 1 adjoined for any hereditary radical.

Theorem 5.2.3 *Let R be a G-graded ring with unity, G an infinite group. Then for any hereditary radical, λ, $\lambda((R, R\#P_G)) = (\lambda(R) \cap \lambda_{ref}(R), \lambda(R\#P_G))$.*

Proof: Take $z = (x, y) \in \lambda((R, R\#P_G))$. Now $y = \Sigma_{g\in G} y(g)\#p_g$. Since G is infinite there is an $h \in G$ with $y(h) = 0$. Multiply z on the right by $(0, 1\#p_h) \in (0, R\#P_G)$, to obtain $(0, x\#p_h)$. It is clear that $(0, x\#p_h)$ is in $(0, R\#P_G)$, and since $\lambda((R, R\#P_G))$ is an ideal, $(0, x\#p_h) \in \lambda((R, R\#P_G)) \cap (0, R\#P_G) = \lambda((0, R\#P_G)) = (0, \lambda_{ref}(R)\#P_G)$. This means that $x \in \lambda_{ref}(R)$.

There is a finite set, H, of $h \in G$ such that $y(\sum_{h\in H} 1\#p_h) = y$. Now suppose we multiply z on the right by $\Sigma_{h\in H}(0, 1\#p_h)$. Then we get $\Sigma_{h\in H}(0, x\#p_h) + (0, y)$, which is in $(0, \lambda_{ref}(R)\#P_G)$. We know that $x \in \lambda_{ref}(R)$, so $a = \Sigma_{h\in H}(0, x\#p_h) \in (0, \lambda_{ref}(R)\#P_G)$. And since $(0, \lambda_{ref}(R)\#P_G)$ is an ideal, $a + (0, y) - a = (0, y) \in (0, \lambda_{ref}(R)\#P_G)$.

So, in combination with Lemma 5.2.2, $\lambda((R, R\#P_G)) \subseteq (\lambda(R) \cap \lambda_{ref}(R), \lambda(R\#P_G))$.

For simplicity, let $I = \lambda(R) \cap \lambda_{ref}(R)$. It is clear that $I \lhd \lambda(R)$, and thus since λ is hereditary, $I \in \lambda$. We can also see that $I \cong (I, \lambda(R\#P_G))/(0, \lambda(R\#P_G))$. This implies that $(I, \lambda(R\#P_G)) \in \lambda$. And by Lemma 5.2.1 we see that it is also an ideal of $(R, R\#P_G)$, so hence is contained in $\lambda((R, R\#P_G))$. Hence we have equality. □

Corollary 5.2.4 *Let R be a G-graded ring with unity. Then the Jacobson radical of $(R, R\#P_G)$ is $(J(R) \cap J_G(R), J_G(R)\#P_G)$.*

Proof: Remember that $J(R\#P_G) = J_G(R)\#P_G$. □

In Section 4.1 we noted that for an A-radical, λ, $\lambda((R, R\#P_G)) = (\lambda(R), \lambda(R)\#P_G)$. Using the same reasoning, we have that $\lambda(R\#P_G) = \lambda(R)\#P_G$. Thus we get that $\lambda(R) = \lambda_{ref}(R)$. Hence, for any A-radical, the statement in Theorem 5.2.3 holds. In Example 4.1.2, we saw that an A-radical need not be hereditary, so the question becomes whether or not Theorem 5.2.3 holds for all radicals. The following example shows that it does not.

Example 5.2.5 Let R be a commutative ring with unity. Then $R[x]$ is a $\mathbf{Z}$-graded ring. Now form the smash product, $R[x]\#P_{\mathbf{Z}}$.

First let us show that $R[x]\#P_{\mathbf{Z}}$ has no ideals with identity. Assume that I is an ideal of $R[x]\#P_{\mathbf{Z}}$ that has an identity, e. Now $e = \sum_{i=1}^{n} f_i(x)\#p_{m_i}$. Let $m < \min\{m_1, \cdots, m_n\}$ and then consider $(g(x)\#p_m)e$ where $0 \neq g(x)\#p_m \in I$. Such elements $g(x)\#p_m$ exist since, for example, $f_1(x)\#p_{m_1} \in I$, $f_1(x)x^{m_1-m}\#p_m \in I$. But

$$(g(x)\#p_m)e = \sum g(x)f_i(x)_{m-m_i}\#p_{m_i}$$

and $m - m_i < 0$ for all i, and so $g(x)\#p_m = (g(x)\#p_m)e = 0$.

Now let $I_0 = \{R[x]\#P_{\mathbf{Z}}\}$, and define $I_{i+1} = \{S|S \text{ is an ideal of a ring in } I_i\}$. Let $\mathcal{I} = \bigcup I_i$. If S is in $\mathcal{I}$, then we will show that S does not have an identity.

Assume we have a chain of ideals, $A \lhd J_n \lhd \cdots \lhd J_1 \lhd R[x]\#P_G$, where A has an identity, e. Then e is in every ideal in the chain. So we have $er \in J_1$ for any $r \in R[x]\#P_{\mathbf{Z}}$, and then $e(er) \in J_2$. But $e(er) = (ee)r = er$. So, we can get that $er \in A$. Therefore A is an ideal of $R[x]\#P_{\mathbf{Z}}$, and does not have an identity.

Now the set $\mathcal{I}$ has the property that if $R \in \mathcal{I}$ and $S \lhd R$, then $S \in \mathcal{I}$. We define a class of rings $U_{\mathcal{I}} = \{R|R/J \in \mathcal{I} \iff J = R\}$, and claim that $U_{\mathcal{I}}$ is a radical class.

First, let $R \in U_{\mathcal{I}}$ and let R/A be a homomorphic image of R. Now take $B \lhd R/A$ and form $\frac{R/A}{B}$, which is a homomorphic image of R/A. This is again in turn a homomorphic image of R and is isomorphic to R/C for some ideal C of R. Now $R/C \in \mathcal{I}$ if and only if $C = R$. So B must be equal to R/A. Hence R/A is in $U_{\mathcal{I}}$.

Next, let $A \in U_{\mathcal{I}}$ and $R/A \in U_{\mathcal{I}}$. Then take $R/B \in \mathcal{I}$. If $A \subseteq B$ then $A \lhd B$ and then $R/B \cong \frac{R/A}{B/A}$, implying that $B/A = R/A$. Thus $B = R$.

If $A \not\subseteq B$ then $(A+B)/B \cong A/(A \cap B)$ is an ideal of R/B, and hence is in $\mathcal{I}$. But that would mean that $A \subseteq B$ which contradicts the assumption. Therefore R/B must not be in $\mathcal{I}$ except when $B = R$. So $R \in U_{\mathcal{I}}$.

Finally, let $J_1 \lhd J_2 \lhd \cdots$ be an ascending chain of ideals with each $J_i \in \mathcal{I}$, and let $J = \bigcup J_i$. Now take $J/A \in \mathcal{I}$. Note also that $J_i \lhd J$ for all i. Thus we get that $J_i/(J_i \cap A) = (J_i + A)/A$ is an ideal of J/A, and is therefore in $\mathcal{I}$. That means that $J_i \subseteq A$. This holds for all i. Hence $J \subseteq A$, implying that $A = J$. Hence $J \in U_{\mathcal{I}}$. Therefore $U_{\mathcal{I}}$ is a radical class by Definition 4.1.1.

Since no ring in $\mathcal{I}$ has a unity, no ring with unity can be mapped onto a ring in $\mathcal{I}$. Thus if S has unity, then $U_{\mathcal{I}}(S) = S$, where $U_{\mathcal{I}}(S)$ is the largest ideal of S in the class $U_{\mathcal{I}}$ as in Definition 4.1.1. In particular, $U_{\mathcal{I}}((R[x], R[x]\#P_{\mathbf{Z}})) = (R[x], R[x]\#P_{\mathbf{Z}})$. But $U_{\mathcal{I}}(R[x]\#P_{\mathbf{Z}}) = 0$, since $R[x]\#P_{\mathbf{Z}}$ is in $\mathcal{I}$. So $(U_{\mathcal{I}})_{ref}(R[x]) = 0$. Thus

$$(U_{\mathcal{I}}(R[x]) \cap (U_{\mathcal{I}})_{ref}(R[x]), U_{\mathcal{I}}(R[x]\#P_{\mathbf{Z}})) = 0.$$

This shows that Theorem 5.2.3 does not hold for all radicals. †

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