THE HEAT KERNEL ON NONCOMPACT RIEMANN SURFACES

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Dedication

To my parents,

who have always supported me,

no matter what I did,

or how long it took.

Abstract

The heat equation has the form

$$\left(\Delta + \frac{\partial}{\partial t}\right) u(\mathbf{x}, t) = 0,$$
$$u(\mathbf{x}, 0) = f(\mathbf{x}),$$

where, Δ is the Laplacian $(d + d^*)^2$ and $f(\mathbf{x})$ is the inital condition. In the case of functions on $\mathbb{R}^n$, with Cartesian coordinates, $\Delta = -\sum_1^n \frac{\partial^2}{\partial x_i^2}$, which is $-\nabla^2$. This gives the usual form of the heat equation $\nabla^2 u = \partial_t u$. When we consider the heat equation on a manifold, the form of the Laplacian depends on the metric. We discuss the heat equation on differential forms and we will find the solution operator, $e^{-t\Delta}$, for the heat equation on the hyperbolic plane, H^2, expressing it in terms of the Green's function, also known as the heat kernel. We will also show how to find the solution operator for the heat equation on quotients of the hyperbolic plane based on the example of the flat torus as a quotient of the Euclidean plane.

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Table of Contents

List of Tables

List of Figures

Chapter 1
Introduction

1.1 Introduction

In the early nineteenth century, Joseph Fourier published his work on the conduction of heat, *Théorie analytique de la chaleur* or *The Analytical Theory of Heat*. In this work, Fourier expresses what will later be called the heat kernel for the one- and three-dimensional Euclidean spaces [24, §383-4]. The two-dimensional case is not given, though there is enough information to formulate the heat kernel for that case.

Turning to surfaces, in particular, the hyperbolic plane, the heat kernel for functions was found by H. P. McKean [35] in 1970. An explicit formula for the heat kernel in hyperbolic space is given by Grigor'yan and Noguchi [28]. For more information about heat kernels on scalar functions, we direct you to the introduction of another paper [27] by Grigor'yan.

In a paper published in 1988, Ingolf Buttig proved the existence and uniqueness of a global heat kernel for functions on manifolds of bounded geometry [6]. In that same paper Buttig conjectured that there was a unique global heat kernel for differential forms on such manifolds. Three years later, Buttig together with Jürgen Eichhorn provided a proof of that conjecture [7].

We are concerned with the construction of the heat kernel for differential forms in the particular case of the hyperbolic plane because the hyperbolic plane is the universal cover for nearly all Riemann surfaces. This will give a method to find the heat kernel for those Riemann surfaces. An expression for the heat kernel for functions is known, [9, 11], but no similar result is known for the differential forms case. Following methods used by Chavel in [9], and properties of the heat kernel proven by Buttig and Eichhorn in [7], we will provide an expression for the heat kernel for differential forms on the hyperbolic plane. We will also demonstrate a relationship between the heat kernel for functions and the heat kernel for differential forms. Since we are focusing on the hyperbolic plane, we are considering a two-dimensional manifold and thus we will be considering differential 1-forms, sometimes called pfaffian forms [34], or 1-form fields [40]. Forms of degree 0 and 2, which are Hodge-dual, are isomorphic to functions, in which case the problem has already been extensively studied.

After finding an expression for the heat kernel for differential forms on the hyperbolic plane, we will show how that can be extended to give the heat kernel on quotients of the hyperbolic plane by isometry groups.

In Chapter 1 we introduce the terminology and properties of differential forms, and extend the idea of the Dirac delta function to differential forms. In Chapter 2 the heat equation and heat kernel are given, focusing on the case of the Euclidean plane, in both rectangular and polar forms. This is done to give a model on which to base the derivation of the heat kernel on the hyperbolic plane. Chapter 3 shows the details of the construction of the heat kernel for differential forms on the hyperbolic plane, and Chapter 4 shows how we can extend the heat kernel from a covering space to give the heat kernel on the underlying surface.

Throughout this manuscript we will use "†" to note the end of an example, "□"

to note the end of a remark, and "■" to note the end of a proof.

1.2 Manifolds

We will be working with Riemann surfaces, which are examples of two-dimenional manifolds. An n-dimensional manifold is a space which is locally homeomorphic to $\mathbb{R}^n$. For a complete definition and discussion, we refer to [10].

Let M be an n-dimensional manifold, and suppose that on M a symmetric quadratic form, $g_{ij}(\mathbf{x})$, is given. It is conventional to define ds, the line element, by

$$(ds)^2 = g_{ij}(\mathbf{x})dx^i dx^j,$$

where dx^i are the differentials of the local coordinates. Note that we are using the Einstein summation convention. If the quadratic form is positive definite, meaning for a tangent vector $\mathbf{v}$ at the point $\mathbf{x}$, $g_{ij}(\mathbf{x})v^i v^j \geq 0$ with equality if and only if $\mathbf{v} = 0$, we say it is a *Riemannian metric*. A manifold with a Riemannian metric is called a *Riemannian manifold*.

Next, let us consider what happens to the metric under a change of coordinates. Given a set of coordinates, x^i in an open neighbourhood, possibly all of M, of a point $\mathbf{x}$, and another set of coordinates, $x^{i'}$ in the same region, there is a map from one set of coordinates to the other, so that $x^i = x^i(x^{i'})$. Now we define $J^i_{i'} := \frac{\partial x^i}{\partial x^{i'}}$ and we say the coordinate transformation is non-singular if the determinant, $|J^i_{i'}|$ is nowhere zero.

Since the metric is meant to encapsulate the notion of distance, it is reasonable to require that ds^2 does not change under a change of variables. From [44] we learn that $dx^i = J^i_{i'} dx^{i'}$. Thus

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j \\ &= g_{ij} J^i_{i'} dx^{i'} J^j_{j'} dx^{j'} \\ &= g_{ij} J^i_{i'} J^j_{j'} dx^{i'} dx^{j'} \end{aligned}$$

So, from these equations we see that the metric transforms in the following manner:

$$g'_{i'j'}(\mathbf{x}') = g_{ij}(\mathbf{x}) J^i_{i'} J^j_{j'}.$$

Example 1.2.1 Consider the real plane, $\mathbb{R}^2$ under the usual Cartesian coordinates, $x^1 = x$ and $x^2 = y$, and the coordinate transformation to polar coordinates, $x'^1 = r$ and $x'^2 = \theta$. We know that the transformation maps are $x = r\cos\theta$ and $y = r\sin\theta$. This transformation is singular at one point, the origin. The components $J^i_{i'}$ are given by

$$J^i_{i'} = J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

If we consider the metrics as matrices, G and G', and $J^i_{i'}$ also as a matrix, with the superscript as the row index, we see that $G' = J^T G J$, where G is the identity matrix. We then see that

$$G' = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix},$$

which is the Euclidean metric in polar coordinates form. Note that this change of variables is singular when $r = 0$. †

While Riemann surfaces do have complex structure, as a 1-dimensional complex manifold, we will not need that for the work we are to do. For further information about the use of the complex structure of Riemann surfaces, see Farkas and Kra [21].

1.3 Differential Forms

On an n-dimensional Riemannian manifold, M, at each point, $\mathbf{x}$, we are given a tangent space, which is an n-dimensional vector space with basis vectors $\partial_{x^1}|_{\mathbf{x}}, \cdots, \partial_{x^n}|_{\mathbf{x}}$. The tangent space at $\mathbf{x}$ is denoted $T_{\mathbf{x}}(M)$. For further details, see Darling [10]. The tangent bundle over M is a $2n$-dimensional manifold, denoted $T(M)$, is the collection of all tangent spaces, parametrized by the points, $\mathbf{x}$, on the manifold, together with smooth transition functions that take one tangent space into the next.

Locally, the cotangent space is dual to the tangent space and is denoted $T^*_{\mathbf{x}}(M)$. An element of the cotangent space is a linear functional, $L : T_{\mathbf{x}}(M) \to \mathbb{R}$. The basis of this space consists of the maps $dx^1, \ldots, dx^n$. By definition, the two bases interact in the following manner, within a coordinate patch:

$$dx^i(\partial_{x^j}|_{\mathbf{x}}) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

The cotangent bundle is defined similar to the above and is denoted $T^*(M)$. Sections of $T^*(M)$, that is maps from M to $T^*(M)$, are called 1-forms. [11]

On the cotangent space we can define an *exterior product* called the *wedge product*. The wedge product obeys the following rule:

$$dx^i \wedge dx^j + dx^j \wedge dx^i = 0.$$

The wedge product is associative, and a product of the form

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

is called a k-form. We also have a space of 0-forms, which are isomorphic to the space of functions over the manifolds, M, which can, depending on the application, be assumed to be $C^\infty(M)$ or $C_0^\infty(M)$ or $L^2(M)$. The space of k-forms with real coefficients is denoted $\bigwedge^k T^*(M)$, or $C_0^\infty(\bigwedge^k T^*M)$ [8]

These spaces are finite dimensional at each point, with real dimension $\binom{n}{k}$. We denote the basis elements of $\bigwedge^k T_{\mathbf{x}}^*(M)$ by dx^I, where $I = \{i_i, \cdots, i_k\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We choose also an n-form, denoted $dvol(\mathbf{x}) := \sqrt{g(\mathbf{x})}dx^1 \wedge \cdots \wedge dx^n$ which gives an orientation to the manifold and is used in integration over the manifold.

The *exterior differential operator*, d, takes k-forms to $(k+1)$-forms, that is

$$d : \bigwedge^k T^*(M) \to \bigwedge^{k+1} T^*(M).$$

We will define this operator recursively:

$$df(\mathbf{x}) = \sum \left[\partial_{x^i} f(\mathbf{x})\right] dx^i,$$

and

$$d\left[f^I(\mathbf{x}) dx^I\right] = \left[df^I(\mathbf{x})\right] \wedge dx^I.$$

We will assume that the functions $f^I(\mathbf{x})$ are sufficiently smooth. Because mixed partial derivatives are equal, it follows that $dd\omega = 0$ for any k-form ω. This gives a long exact sequence

$$0 \longrightarrow \bigwedge^0 T^*(M) \xrightarrow{d} \bigwedge^1 T^*(M) \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^n T^*(M) \longrightarrow 0.$$

This is sometimes called the *DeRham complex*.

Example 1.3.1 Let us consider the exterior derivative on $\mathbb{R}^3$. Its application to $0, 1,$ and 2-forms can be identified with the classical vector calculus operations of div, grad, and curl in the following manner [10]:

$$\begin{aligned} df &= f_x dx + f_y dy + f_z dx \\ &\sim (f_x, f_y, f_z) = \nabla f \end{aligned}$$

$$\begin{aligned} d(f^1 dx + f^2 dy + f^3 dz) &= (f^3_y - f^2_z) dy \wedge dz + (f^1_z - f^3_x) dz \wedge dx + (f^2_x - f^1_y) dx \wedge dy \\ &\sim (f^3_y - f^2_z, f^1_z - f^3_x, f^2_x - f^1_y) = \operatorname{curl}(f^1, f^2, f^3) \end{aligned}$$

$$\begin{aligned} d(f^1 dy \wedge dz + f^2 dz \wedge dx + f^3 dx \wedge dy) &= (f^1_x + f^2_y + f^3_z) dx \wedge dy \wedge dz \\ &\sim (f^1_x + f^2_y + f^3_z) = \operatorname{div}(f^1, f^2, f^3) \end{aligned}$$

†

This definition of the exterior differential operator appears to be restricted to a single coordinate patch. There is another, equivalent, definition which is global, using covariant derivatives. The covariant derivative of a 1-form field, compatible with the Riemannian metric can be written:

$$f_{\alpha;\beta} := f_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} f_{\gamma}.$$

The notation, $f_{\alpha,\beta}$, indicates the usual partial derivative in the local coordinate system. The Christoffel symbol compatible with the Riemannian metric is,

$$\Gamma^{\gamma}_{\alpha\beta} := \frac{1}{2} g^{\gamma\nu} \left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} \right).$$

It is clearly symmetric in the lower indicies, so

$$f_{\alpha;\beta} - f_{\beta;\alpha} = f_{\alpha,\beta} - f_{\beta,\alpha}.$$

Using the covariant derivative to define the exterior differential operator,

$$df_\alpha dx^\alpha := f_{\alpha;\beta} dx^\beta \wedge dx^\alpha,$$

and isolating a particular differential form, ie. $dx^a \wedge dx^b$, it is clear that the Christoffel terms cancel, leaving only the local partial derivatives. Thus the two definitions are equivalent.

As we mentioned previously, the spaces $\bigwedge^k T^*_{\mathbf{x}}(M)$ are finite dimensional, with dimension $\binom{n}{k}$. This means that $\bigwedge^k T^*_{\mathbf{x}}(M)$ and $\bigwedge^{n-k} T^*_{\mathbf{x}}(M)$ have the same dimension. It is well-known that finite dimensional vectors spaces of the same dimension are isomorphic, and we can define the isomorphism by how it acts on the basis vectors. The standard isomorphism, called the Hodge star isomorphism, denoted $* : \bigwedge^k T^*_{\mathbf{x}}(M) \to \bigwedge^{n-k} T^*_{\mathbf{x}}(M)$, is given by the formula in [18]:

$$*dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \frac{\sqrt{g}}{(n-k)!} \epsilon^{i_1 \cdots i_k}{}_{\alpha_{k+1} \cdots \alpha_n} dx^{\alpha_{k+1}} \wedge \cdots \wedge dx^{\alpha_n} \tag{1.3.1}$$

where

$$\epsilon_{i_1 \cdots i_n} = \begin{cases} 0 & \text{repeated index} \\ 1 & \text{even permutation} \\ -1 & \text{odd permutation} \end{cases}$$

and $\epsilon_{i_1 \cdots i_n}$ is the permutation symbol. We lower an index in the following manner: $\frac{g_{\alpha i_1}}{g} \epsilon^{\alpha}{}_{i_2 \cdots i_n} = \epsilon_{i_1 \cdots i_n}$. Also, we have

$$* * dx^{i_1} \wedge \cdots \wedge dx^{i_k} = (-1)^{k(n-k)} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Example 1.3.2 Since we will be considering surfaces for the majority of this thesis, let us examine more closely the Hodge star isomorphism for a 2-dimensional manifold. Let the metric for the manifold have components g_{ij} in some chart, with $g_{12} = g_{21}$ and assume that $g = \det g_{ij}$ is not identically zero. Then, from (1.3.1), we have

$$\begin{aligned} *1 &= \sqrt{g}dx^1 \wedge dx^2 && (1.3.2) \\ *dx^1 &= \frac{1}{\sqrt{g}}\left[g_{12}dx^1 + g_{22}dx^2\right] && (1.3.3) \\ *dx^2 &= \frac{-1}{\sqrt{g}}\left[g_{11}dx^1 + g_{12}dx^2\right] && (1.3.4) \\ *dx^1 \wedge dx^2 &= \frac{1}{\sqrt{g}} && (1.3.5) \end{aligned}$$

If the metric is diagonal, that is $g_{12} = g_{21} = 0$, then the isomorphism simplifies greatly. †

Example 1.3.3 If we consider the Hodge star operator on differential forms in $\mathbb{R}^3$, there is another vector calculus operation which we can define. For 1-forms, we have the following identity:

$$\begin{aligned} &*[(f^1dx + f^2dy + f^3dz) \wedge (g^1dx + g^2dy + g^3dz)] \\ &= (f^2g^3 - f^3g^2)dx + (f^3g^1 - f^1g^3)dy + (f^1g^2 - f^2g^1)dz. \end{aligned}$$

This corresponds to the $(f^1, f^2, f^2) \times (g^1, g^2, g^3)$ [10]. †

If we form the product $\omega(\mathbf{x}) \wedge *\omega(\mathbf{x})$ we get a multiple of the n-form $dx^1 \wedge \cdots \wedge dx^n$. To see this, consider $\omega(\mathbf{x}) = f(\mathbf{x})dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

$$\begin{aligned} \omega \wedge *\omega &= f(\mathbf{x})dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge f(\mathbf{x})\frac{\sqrt{g}}{(n-k)!}\epsilon^{i_1\cdots i_k}{}_{\alpha_{k+1}\cdots\alpha_n} dx^{\alpha_{k+1}} \wedge \cdots \wedge dx^{\alpha_n} \\ &= f^2(\mathbf{x})\frac{\sqrt{g}}{(n-k)!}\epsilon^{i_1\cdots i_k}{}_{\alpha_{k+1}\cdots\alpha_n} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{\alpha_{k+1}} \wedge \cdots \wedge dx^{\alpha_n} \end{aligned}$$

From this, we define a *local inner product*:

$$\langle \omega(\mathbf{x}), \nu(\mathbf{x}) \rangle \sqrt{g}dx^1 \wedge \cdots \wedge dx^n = \langle \omega(\mathbf{x}), \nu(\mathbf{x}) \rangle * 1 := \omega(\mathbf{x}) \wedge *\nu(\mathbf{x}),$$

for two k-forms ω and ν. If ω and ν are of different degree, then the inner product is defined to be 0.

Having introduced this inner product, we next consider normalizing the basis vectors (forms) for $\bigwedge^k(M)$, so as to give them unit length with respect to the local inner product. Since we consider these as vector spaces, having a normalized basis would be a reasonable step, though as we will see later, it is not always convenient. We will define, as usual, the normalized basis vector, $\overline{dx^I}$, as

$$\overline{dx^I} := \frac{1}{\|dx^I\|} dx^I = \left\langle dx^I, dx^I \right\rangle^{-\frac{1}{2}} dx^I,$$

with $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$. This normalized basis is sometimes referred to as an orthonormal coframe field [10].

Example 1.3.4 Let us consider the two dimensional case, with symmetric metric g_{ij}. We will denote, as usual, $g = \det(g_{ij})$. To start, we know the 0-form, 1, has unit length, since $*1 = \sqrt{g} dx^1 \wedge dx^2$, according to equation (1.3.2). Similarly, from equation (1.3.5), we have

$$dx^1 \wedge dx^2 \wedge *(dx^1 \wedge dx^2) = dx^1 \wedge dx^2 \wedge \frac{1}{\sqrt{g}}.$$

Clearly, $\langle dx^1 \wedge dx^2, dx^1 \wedge dx^2 \rangle = g^{-1}$, thus

$$\overline{dx^1 \wedge dx^2} = \sqrt{g} dx^1 \wedge dx^2.$$

For 1-forms, we will consider in detail the case of dx^1, then state the result for dx^2. From equation (1.3.3) we see

$$dx^1 \wedge *dx^1 = dx^1 \wedge \frac{1}{\sqrt{g}} \left[g_{12} dx^1 + g_{22} dx^2 \right] = \frac{g_{22}}{\sqrt{g}} dx^1 \wedge dx^2.$$

Since

$$\left\langle dx^1, dx^1 \right\rangle \sqrt{g} dx^1 \wedge dx^2 = \frac{g_{22}}{\sqrt{g}} dx^1 \wedge dx^2,$$

it is clear that

$$\overline{dx^1} = \sqrt{\frac{g}{g_{22}}} dx^1 = \sqrt{g^{11}} dx^1.$$

Similarly,

$$\overline{dx^2} = \sqrt{\frac{g}{g_{11}}} dx^2 = \sqrt{g^{22}} dx^2.$$

As mentioned before, sometimes it is not convenient to normalize the basis forms. To see this, we note that, unfortunately,

$$\overline{dx^1} \wedge \overline{dx^2} \neq \overline{dx^1 \wedge dx^2},$$

unless the metric is diagonal.

In the two dimensional case, which will be our main concern, if the metric has a singularity, such as the polar coordinates version of the plane, this normalization process could cause some of the forms to vanish. This is illustrated by $\overline{d\theta} = r d\theta$ and the singularity at the origin. †

There is also a *global inner product*, which is the integral of the local inner product, so that

$$\langle \omega(\mathbf{x}), \nu(\mathbf{x}) \rangle_g = \int_M \omega(\mathbf{x}) \wedge *\nu(\mathbf{x}) = \int_M \langle \omega(\mathbf{x}), \nu(\mathbf{x}) \rangle * 1.$$

For information about integration on a manidfold, please see [23, 43]. Again, we define the inner product of forms of different degrees to be zero, and we can see that $dx^I \perp dx^J$ for $I \neq J$. However, if the manifold is infinite in volume, then integration over the manifold must be done in the correct function space. With this in mind, $\left\langle dx^I, dx^I \right\rangle_g$ may not have any meaning. If the functions are, for example, C_c^∞ or L^2, then we can safely use the inner product. Note, that unless the manifold is compact, the k-form dx^I is not in $\bigwedge^k T^*(M)$ since 1 is not compactly supported, nor is it L^2.

Let us define a *co-differential operator*, d^*, as [25]

$$d^*\omega = (-1)^{nk+n+1} * d * \omega,$$

where ω is a k-form on an n-dimensional manifold. Because this involves the Hodge star isomorphism the coderivative, unlike the exterior derivative, depends on the metric. It can be shown that the co-differential operator is the formal adjoint of the exterior derivative with respect to the global inner product when the global inner product exists, which can always be arranged by cutting off the functions f, g outside a compact set. As with the exterior derivative, we will assume that the functions involved are sufficiently smooth.

The co-differential operator is a map from k-forms to $(k-1)$-forms. Like the exterior derivative, $d^*d^*\omega = 0$. Thus we can form another long exact sequence

$$0 \longrightarrow \bigwedge^{n} T^*(M) \xrightarrow{d^*} \bigwedge^{n-1} T^*(M) \xrightarrow{d^*} \cdots \xrightarrow{d^*} \bigwedge^{0} T^*(M) \longrightarrow 0.$$

The co-differential and the exterior derivative are related by the equations

$$*d^*\omega = (-1)^k d * \omega,$$

and

$$*d\omega = (-1)^{k+1} d^* * \omega.$$

where $\omega \in \bigwedge^k T^*(M)$.

Let $D := d + d^*$. This the Dirac operator. It is clear that D is self-adjoint with respect to the global inner product. Also, D^2 takes k-forms to k-forms. To check this, note that $D^2 = dd^* + d^*d$ because the squared terms disappear. We will define the *Laplacian* as $\Delta = D^2$.

We write $\Delta_x^{(k)}$ for the Laplacian on k-forms with respect to the $\mathbf{x}$ space variable. If we are discussing Laplacians for two different forms of the metric, for example cartesian and polar, we will also index the Laplacian with the metric, for example $\Delta^{(i)}_{,g}$ is the Laplacian on i-forms with respect to the metric g_{ij}.

1.4 Dirac Delta Function

Let us just remind ourselves about Dirac delta functions. A brief historical survey of the Dirac delta function can be found in [4, §1.1-2]. Its application to solving differential equations can be found in [33, §5.4].

We will consider the Dirac delta function on the real line, and then generalize. We want a two-parameter generalized function, f, that will obey several conditions: $\int_{\mathbb{R}} f(x,y)dx = 1$ regardless of y; the support of f must be a single point, $x = y$; and the operator $\int_{\mathbb{R}} f(x,y)g(x)dx$ is evaluation of g at the point y.

To make things simpler, we will fix $y = 0$ and consider f as a one-variable function. We will consider the sequence of functions

$$f_n(x) = \begin{cases} \dfrac{n}{2}, & |x| \leq \dfrac{1}{n} \\ \\ 0, & |x| > \dfrac{1}{n} \end{cases}$$

Clearly,

$$\int_{\mathbb{R}} f_n(x)dx = 1$$

and the support of the limit, in the distributional sense, of these functions is

$$\operatorname{supp}\left\{\lim_n f_n\right\} = \bigcap \left[-\frac{1}{n}, \frac{1}{n}\right] = \{0\}.$$

This leaves us only to check that the limit does provide evaluation at the point, $y = 0$. For the function f_n we see

$$\int_{-\infty}^{\infty} f_n(x)g(x)dx = \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} g(x)dx = \frac{n}{2}\left(\frac{2}{n}g(c)\right) = g(c),$$

where $|c| < 1/n$ by the Mean Value theorem. As $n \to \infty$ we see that the integral converges to $g(0)$. We will define

$$\delta(x, 0) := \lim_{n\to\infty} f_n(x),$$

and we extend to arbitrary y by

$$\delta(x, y) = \delta(x - y, 0).$$

We note that $\delta(x, y) = \delta(y, x)$, and we will also comment that the Dirac delta function is not really a function, but a distribution. More information on distributions can be found in [4].

Let us write the derivative of the Dirac delta function, $\partial_x \delta(x, y)$. Technically, as we think of derivative, this expression has no meaning, just familiar symbols juxtaposed. However, if we insert $\partial_x \delta(x, y)$ into the integral against a C_c^1 test function, g, then, using integration by parts we see

$$\int_{\mathbb{R}} \left[\partial_x \delta(x, y)\right] g(x) dx = \delta(x, y) g(x)|_{-\infty}^{\infty} - \int_{\mathbb{R}} \delta(x, y) \partial_x g(x) dx = -\partial_x g(x)\bigg|_y ,$$

since $\delta(x, y) = 0$ for $x \neq y$. We can extend this so that

$$\int_{\mathbb{R}} \left[\partial_x^n \delta(x, y)\right] g(x) dx = (-1)^n \partial_x^n g(x)\bigg|_y ,$$

for a C_c^n function, g.

The Dirac delta function can be extended to $\mathbb{R}^n$ with cartesian coordinates by taking a product of n one-dimensional delta functions, one for each coordinate: [5]

$$\delta(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} \delta(x^i, y^i). \tag{1.4.1}$$

Of course, distributions cannot ordinarily be multiplied together, however in the special case of the Dirac delta function the product is defined. This multi-dimensional delta function behaves in a similar manner to the single dimensional version. [5] So we have:

$$\int_{B_\epsilon(\mathbf{x}_0)} f(\mathbf{x})\delta(\mathbf{x},\mathbf{x}_0)dvol(\mathbf{x}) = f(\mathbf{x}_0)$$

for an open ball in $\mathbb{R}^n$ and f continuous at $\mathbf{x}_0$.

We can also transform the delta function for the case of curvilinear coordinates in $\mathbb{R}^n$. If $\mathbf{x}', \mathbf{x}'_0 \in \mathbb{R}^n$ and we impose orthogonal curvilinear coordinates $(x'^1, \cdots, x'^n)$ and $(x'^1_0, \cdots, x'^n_0)$ respectively, then

$$\delta(\mathbf{x}', \mathbf{x}'_0) = \frac{1}{\Lambda_1 \cdots \Lambda_n} \prod \delta(x'^i, x'^i_0),$$

where $\Lambda_i^2 = \sum_j (\frac{\partial x^j}{\partial x'^i})^2$, with $x^i = x^i(x'^i)$ are the cartesian coordinates of $\mathbf{x}$.

Example 1.4.1 Using polar coordinates in $\mathbb{R}^2$, we have $x = r\cos\theta$ and $y = r\sin\theta$, so $\Lambda_r^2 = \cos^2\theta + \sin^2\theta = 1$ and $\Lambda_\theta^2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2$. Thus the delta function is

$$\delta(r,\theta; r_0,\theta_0) = \frac{1}{r}\delta(r,r_0)\delta(\theta,\theta_0). \tag{1.4.2}$$

†

1.5 Two-Point Forms

When working with single variable integration over a compact interval, if we evaluate a definite integral we get a scalar value. In integral transforms like the Fourier or Laplace transform, we integrate a two-variable function, obtaining a one-variable function. When we integrate forms, we similarly sometimes wish to get a form-valued result. To achieve this we introduce, via tensor products, what we refer to as two-point forms. These are also called double currents by de Rham [12].

How are we going to represent a two-point form? Recall from Section 1.3 that a differential k-form (or a one-point k-form) can be represented as

$$\omega(\mathbf{y}) = \sum_{|I|=k} f^I(\mathbf{y}) dx^I,$$

where the dx^I are basis elements of $\bigwedge^k T^*(M)$, and the f^I are in some appropriate function space. If we had another point, $\mathbf{z} \in M$, then we say that an arbitrary two-point (k,p)-form (or just a two-point k-form if $p = k$) is written

$$\omega(\mathbf{y}, \mathbf{z}) = \sum_{\substack{|I|=k \\ |J|=p}} f^{I,J}(\mathbf{y}, \mathbf{z}) dx^I \otimes dx^J \in \bigwedge^k T^*(M) \otimes \bigwedge^p T^*(M).$$

We can extend the usual operators on one-point forms to two-point forms. The extension is quite natural, for example, $d_{\mathbf{y}}\omega(\mathbf{y}, \mathbf{z})$ would act on the $dx^I(\mathbf{y})$ portion with the partial derivatives taken with respect to the variable $\mathbf{y}$, while the $\mathbf{z}$ terms would be held constant. We can define the codifferential operator and the Hodge star in a similar manner.

Next, we consider the wedge product in relation to two-point forms. When considering $\omega(\mathbf{y}, \mathbf{z}) \wedge_{\mathbf{z}} \nu(\mathbf{z})$ where ω is a two-point form, and ν is a one-point form, then wedge product is applied only to the appropriate portion of ω and ν. For example, $\left[dx^I(\mathbf{y}) \otimes dx^J(\mathbf{z})\right] \wedge_{\mathbf{z}} dx^K(\mathbf{z}) = dx^I(\mathbf{y}) \otimes \left[dx^J(\mathbf{z}) \wedge dx^K(\mathbf{z})\right]$.

When we consider the global inner product of a two-point form and a one-point form,

$$\langle \omega(\mathbf{y}, \mathbf{z}), \nu(\mathbf{z}) \rangle_g = \int_M \omega(\mathbf{y}, \mathbf{z}) \wedge_{\mathbf{z}} *_{\mathbf{z}} \nu(\mathbf{z}),$$

we must first of all ensure that if ν is a k-form, then ω is a member of $\bigwedge^p T^*_{\mathbf{y}}(M) \otimes \bigwedge^k T^*_{\mathbf{z}}(M)$. This ensures we will get an n-form in the integral. Second, when the integration is performed, the result will be a $(p,0)$-form $\xi(\mathbf{y}) \otimes 1_{\mathbf{z}}$ which we will generally identify with the one-point p-form $\xi(\mathbf{y})$.

Remark 1.5.1 Now, the question may arise, can we define $\omega(\mathbf{y},\mathbf{y}) \wedge_{\mathbf{y}} \nu(\mathbf{y})$? This would be of concern if we were considering a two-point form over the diagonal in $M \times M$. For most cases, this would be ill-defined, for example,

$$\left[dx^1(\mathbf{y}) \otimes dx^2(\mathbf{y})\right] \wedge dx^2(\mathbf{y})$$

would be either

$$\left[dx^1(\mathbf{y}) \wedge dx^2(\mathbf{y})\right] \otimes dx^2(\mathbf{y})$$

or

$$dx^1(\mathbf{y}) \otimes \left[dx^2(\mathbf{y}) \wedge dx^2(\mathbf{y})\right] = dx^1(\mathbf{y}) \otimes 0$$

depending on whether the wedge was applied to the first or the second term. We can resolve this difficulty by performing the wedge product first, with $\mathbf{x} \neq \mathbf{y}$ using either the wedge with respect to either $\mathbf{x}$ or $\mathbf{y}$ as the case may warrant, then evaluate the result at $\mathbf{x} = \mathbf{y}$.

However, if ω were a diagonal two-point form, that is, of the form $\omega(\mathbf{y},\mathbf{z}) = \sum f_I(\mathbf{y},\mathbf{z}) dx^I(\mathbf{y}) \otimes dx^I(\mathbf{z})$ there would be no ambiguity as we consider $dx^I(\mathbf{y}) \otimes dx^J(\mathbf{y}) \cong dx^J(\mathbf{y}) \otimes dx^I(\mathbf{y})$. $\square$

1.6 Dirac Delta Forms

In this manuscript, we are primarily interested in hyperbolic space and its quotients, which are smooth Riemannian manifolds with nice symmetry properties. Since the Dirac delta function is supported only at one point, we merely need to concern ourselves with one coordinate patch and not with transfer functions.

Given an n-manifold, M, with diagonal metric, g_{ij}, we are able to find the Dirac delta k-form in terms of the Dirac delta function (or the Dirac delta 0-form), as follows.

We desire that the Dirac delta k-form, denoted $\delta_k(\mathbf{x}, \mathbf{y})$, have the following property:

$$\langle \delta_k(\mathbf{x}, \mathbf{y}), \omega(\mathbf{y}) \rangle_g = \omega(\mathbf{x}), \tag{1.6.1}$$

where $\omega(\mathbf{x})$ is a k-form. Note that δ_k is a two-point form. Because we want this property to hold for all k-forms, ω, the first restriction on δ_k is that if

$$\delta_k(\mathbf{x}, \mathbf{y}) = \sum_{|I|=k} \sum_{|J|=k} A_{I,J}(\mathbf{x}, \mathbf{y}) dx^I(\mathbf{x}) \otimes dx^J(\mathbf{y}),$$

then the value of $A_{I,J}$ for $I \neq J$, does not matter, since the inner product of the delta function with $\omega(\mathbf{y}) dx^I(\mathbf{y})$ would yield zero, instead of satisfying (1.6.1). So we will set $A_{I,J} = 0$ for $I \neq J$. In this case we say that δ_k is diagonal, meaning

$$\delta_k(\mathbf{x}, \mathbf{y}) = \sum_{|I|=k} A_{I,I}(\mathbf{x}, \mathbf{y}) dx^I(\mathbf{x}) \otimes dx^I(\mathbf{y}). \tag{1.6.2}$$

The next restriction we place on δ_k is that under integration, the order of the variables is unimportant, that is, $\int \delta_k(\mathbf{x}, \mathbf{y}) \cdot d\mathbf{x} = \int \delta_k(\mathbf{y}, \mathbf{x}) \cdot d\mathbf{x}$. Because the Dirac delta form is diagonal, this restriction passes to the $A_{I,I}$.

The inner product used in (1.6.1) is the global inner product

$$\langle \omega(\mathbf{x}), \nu(\mathbf{x}) \rangle_g := \int_M \omega(\mathbf{x}) \wedge *\nu(\mathbf{x}),$$

where ω and ν have the same degree. It is clear that $dx^I \perp dx^J$ for $I \neq J$. Thus for k-forms, we can write

$$\langle \delta_k(\mathbf{x}, \mathbf{y}), \omega(\mathbf{y}) \rangle_g = \sum_{|I|=k} \left\langle A_{I,I}(\mathbf{x}, \mathbf{y}) dx^I(\mathbf{x}) \otimes dx^I(\mathbf{y}), f_I(\mathbf{y}) dx^I(\mathbf{y}) \right\rangle_g,$$

where $\omega(\mathbf{x}) = \sum_{|I|=k} f_I(\mathbf{x}) dx^I(\mathbf{x})$.

Let us assume that δ_0 is known and satisfies

$$\begin{aligned}
\langle \delta_0(\mathbf{x},\mathbf{y})1_{\mathbf{x}} \otimes 1_{\mathbf{y}}, f(\mathbf{y})1_{\mathbf{y}}\rangle_g &= \int_M \delta_0(\mathbf{x},\mathbf{y})1_{\mathbf{x}} \otimes 1_{\mathbf{y}} \wedge_{\mathbf{y}} *_{\mathbf{y}} f(\mathbf{y})1_{\mathbf{y}} \\
&= \int_M \delta_0(\mathbf{x},\mathbf{y}) f(\mathbf{y}) 1_{\mathbf{x}} \otimes dvol(\mathbf{y}) \\
&= f(\mathbf{x})1_{\mathbf{x}}. \qquad (1.6.3)
\end{aligned}$$

Now let us see if we can find a distribution, tentatively denoted δ_k, satisfying (1.6.1), using the k-form $f_I(\mathbf{x})dx^I$, where $I = i_1 \ldots i_k$. This means that we are trying to find appropriate $A_{I,I}(\mathbf{x},\mathbf{y})$. We desire that

$$\left\langle A_{I,I}(\mathbf{x},\mathbf{y})dx^I \otimes dy^I, f(\mathbf{y})dy^I \right\rangle_g = f(\mathbf{x})dx^I,$$

so let us expand this in integral form:

$$\begin{aligned}
&\int_M A_{I,I}(\mathbf{x},\mathbf{y})dx^I \otimes dy^I \wedge * f(\mathbf{y})dy^I \\
=&\int_M A_{I,I}(\mathbf{x},\mathbf{y}) f(\mathbf{y}) \frac{\sqrt{g(\mathbf{y})}}{(n-k)!} \epsilon^{i_1 \cdots i_k}{}_{\alpha_{k+1}\cdots\alpha_n} dx^I \otimes dy^{i_1} \wedge \cdots \wedge dy^{i_k} \wedge dy^{\alpha_{k+1}} \wedge \cdots \wedge dy^{\alpha_n} \\
=&\int_M A_{I,I}(\mathbf{x},\mathbf{y}) f(\mathbf{y}) \sqrt{g(\mathbf{y})} g^{i_1\alpha_1} \cdots g^{i_k\alpha_k} \epsilon_{\alpha_1\cdots\alpha_k i_{k+1}\cdots i_n} dx^I \otimes dy^{i_1} \wedge \cdots \wedge dy^{i_n} \\
=&\int_M A_{I,I}(\mathbf{x},\mathbf{y}) f(\mathbf{y}) \sqrt{g(\mathbf{y})} \det(g^{ab}|_{i_1\cdots i_k}) \epsilon_{i_1\cdots i_n} dx^I \otimes dy^{i_1} \wedge \cdots \wedge dy^{i_n} \\
=&\int_M A_{I,I}(\mathbf{x},\mathbf{y}) f(\mathbf{y}) \det(g^{ab}|_{i_1\cdots i_k}) dx^I \otimes dvol(\mathbf{y})
\end{aligned}$$

The term $\det(g^{ab}|_{i_1\cdots i_k})$ is the generalized minor of g^{ab} with the $i_1, \cdots, i_k$ rows and columns retained. For example, if $g^{ab} = a + b$ for $a, b \in \{1, 2, 3, 4\}$, then

$$g^{ab}|_{24} = \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}.$$

From Equation (1.6.3) we see, if we choose

$$A_{I,I}(\mathbf{x},\mathbf{y}) = \frac{1}{\det(g^{ab}|_{i_1\cdots i_k})(\mathbf{y})} \delta_0(\mathbf{x},\mathbf{y}), \qquad (1.6.4)$$

we have the desired property, that

$$\left\langle \frac{1}{\det(g^{ab}|_{i_1\cdots i_k})(\mathbf{y})}\delta_0(\mathbf{x},\mathbf{y})dx^I \otimes dy^I, f(\mathbf{y})dy^I \right\rangle_g = f(\mathbf{x})dx^I.$$

So

$$\delta_k(\mathbf{x},\mathbf{y}) = \sum_{|I|=k} \frac{1}{\det(g^{ab}|_{i_1\cdots i_k})(\mathbf{y})}\delta_0(\mathbf{x},\mathbf{y})dx^I(\mathbf{x}) \otimes dx^I(\mathbf{y}).$$

Example 1.6.1 To illustrate the Dirac delta forms, we will consider two cases: Cartesian coordinates and polar coordinates in $\mathbb{R}^2$.

In the Cartesian case, the metric is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and its inverse, g^{ij}, is also the identity matrix. From equation (1.4.1) we know that the 0-form delta function is

$$\delta_0(\mathbf{x},\mathbf{y}) = \delta(x^1, y^1)\delta(x^2, y^2).$$

Using the formula in equation (1.6.4), we have

$$\delta_1(\mathbf{x},\mathbf{y}) = \delta_0(\mathbf{x},\mathbf{y})\left[dx^1 \otimes dy^1 + dx^2 \otimes dy^2\right]$$

and

$$\delta_2(\mathbf{x},\mathbf{y}) = \delta_0(\mathbf{x},\mathbf{y})\left[dx^1 \wedge dx^2 \otimes dy^1 \wedge dy^2\right].$$

In the polar case, we use equation (1.4.2) to get the 0-form delta function

$$\delta_0(\mathbf{r},\mathbf{s}) = \frac{1}{\rho}\delta(r,\rho)\delta(\theta,\phi),$$

where $\mathbf{r} = (r,\theta)$ and $\mathbf{s} = (\rho,\phi)$. The metric for polar coordinates is

$$g_{ij}(\mathbf{r}) = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

and

$$g^{ij}(\mathbf{r}) = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}.$$

With this information, we can readily write down the 1- and 2-form delta forms:

$$\delta_1(\mathbf{r}, \mathbf{s}) = \delta_0(\mathbf{r}, \mathbf{s}) \left[dr \otimes d\rho + r\rho d\theta \otimes d\phi\right]$$

and

$$\delta_2(\mathbf{r}, \mathbf{s}) = \delta_0(\mathbf{r}, \mathbf{s}) \left[r\rho dr \wedge d\theta \otimes d\rho \wedge d\phi\right].$$

†

Just as the delta function has special properties under differentiation, so do delta forms under the exterior derivative and co-derivative. Using the fact that d and d^* are adjoint operators under the global inner product, we have the following two identities:

1. $\langle d_{\mathbf{y}} \delta_k(\mathbf{x}, \mathbf{y}), \omega(\mathbf{y}) \rangle_g = d^*_{\mathbf{x}} \omega(\mathbf{x})$, for ω a $(k+1)$-form.
2. $\left\langle d^*_{\mathbf{y}} \delta_k(\mathbf{x}, \mathbf{y}), \omega(\mathbf{y}) \right\rangle_g = d_{\mathbf{x}} \omega(\mathbf{x})$, for ω a $(k-1)$-form.

If we change coordinates in a manner that leaves the metric diagonal, then the transformed Dirac delta form will depend only on the transformed metric and Dirac delta function (0-form). It is also important to note that if the metric g_{ij} is diagonal, then $\delta_k = *_{\mathbf{x}} *_{\mathbf{y}} \delta_{n-k}$.

Chapter 2
The Heat Equation

2.1 The Heat Equation and Heat Kernel

To study the classical case of heat flow over a plane with cartesian coordinates, the following partial differential equation is used:

$$(\Delta + \partial_t)u = -u_{x_1x_1} - u_{x_2x_2} + u_t = 0 \tag{2.1.1}$$

with given initital conditions $u(x_1, x_2, 0) = f(x_1, x_2)$. This is called the heat equation with a given initial heat distribution, $f(x_1, x_2)$. The operator, Δ which differentiates with respect to the space variables (x_1 and x_2) is called the Laplacian, and can be written

$$\Delta = -\partial_{x_1x_1} - \partial_{x_2x_2}$$

in the case of cartesian coordinates.

We can solve the heat equation on $\mathbb{R}^2$ using two different methods. Separation of variables is an obvious choice. However, for our purposes, it seems best to determine the Green's function of this equation. This means that we are going to try and find a function $K(\mathbf{x}, \mathbf{y}, t)$ in two space variables, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, and one time variable, which when integrated against the initial condition, $f(\mathbf{x})$, gives the solution of (2.1.1). That is

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^2} K(\mathbf{x}, \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y}. \tag{2.1.2}$$

This function K is called the *heat kernel*. The heat kernel is also the translation of the solution of the heat equation with initial conditions of a Dirac delta function at the origin. [16]

The *solution operator* for the heat equation is written $e^{-t\Delta}$ and is defined by,

$$u(\mathbf{x}, t) = e^{-t\Delta}u(\mathbf{x}, 0).$$

It is clear that $e^{-t\Delta}u(\mathbf{x}, 0)$ satifies the heat equation, with initial conditions $u(\mathbf{x}, 0)$. This is a convenient way of writing (2.1.2), and it is more than merely symbolic. If we substitute $u = e^{-t\Delta}u(\mathbf{x}, 0)$ into the heat equation, we get

$$(\Delta + \partial_t)e^{-t\Delta}u(\mathbf{x}, 0) = \Delta e^{-t\Delta}u(\mathbf{x}, 0) - \Delta e^{-t\Delta}u(\mathbf{x}, 0) = 0$$

just by differentiating the exponential. Also, the semi-group property for the heat kernel is evident: $e^{-t\Delta}(e^{-s\Delta}u(\mathbf{x}, 0)) = e^{-(t+s)\Delta}u(\mathbf{x}, 0)$.

The heat kernel K is a symmetric function in $\mathbf{x}, \mathbf{y}$ and is a solution to the heat equation. That means that

$$\Delta_{\mathbf{x}}K(\mathbf{x}, \mathbf{y}, t) = -\frac{\partial}{\partial t}K(\mathbf{x}, \mathbf{y}, t).$$

We also have the semi-group property [39],

$$K(\mathbf{x}, \mathbf{z}, s + t) = \int_M K(\mathbf{x}, \mathbf{y}, s)K(\mathbf{y}, \mathbf{z}, t)d\mathbf{y},$$

which can also be written $e^{-s\Delta}e^{-t\Delta} = e^{-(s+t)\Delta}$.

To determine the heat kernel we will use integral transforms, in this case, the Fourier integral transform. If we have a function $f(x)$ which satisfies some appropriate conditions, we can write the Fourier transform of f in the following manner [19]:

$$\widehat{f}(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx \tag{2.1.3}$$

and we can undo the transform with the inversion formula:

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(s) e^{iys} ds. \tag{2.1.4}$$

The variable s is sometimes called the transform variable.

Using formulas found in many textbooks and handbooks, for example [19], it can be shown that the Fourier transform of a derivative can be written in terms of the transform of the original function. In our case,

$$\widehat{u_{xx}}(s) = -s^2 \widehat{u}(s).$$

This is demonstrated using integration by parts, assuming suitable properties at infinity. If we perform the Fourier transform twice on (2.1.1), once for each coordinate variable, that is, if we apply the two-variable Fourier transform, we get the following ordinary differential equation:

$$(r^2 + s^2)\widehat{\widehat{u}}(r, s, t) + \widehat{\widehat{u}}_t(r, s, t) = 0.$$

We can solve this exactly, giving

$$\widehat{\widehat{u}}(r, s, t) = \widehat{\widehat{f}}(r, s) e^{-(r^2+s^2)t}.$$

Using the inversion formula (2.1.4) twice will give:

$$\begin{aligned}
u(a, b, t) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{\widehat{f}}(r, s) e^{-(r^2+s^2)t} e^{iar} e^{ibs} dr\ ds \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}^4} f(x, y) e^{-r^2 t} e^{-s^2 t} e^{-ixr} e^{-iys} e^{ira} e^{isb} dx\ dy\ dr\ ds \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(x, y) \int_{\mathbb{R}} e^{-r^2 t} e^{-ir(x-a)} dr \int_{\mathbb{R}} e^{-s^2 t} e^{-is(y-b)} ds\ dx\ dy \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(x, y) \sqrt{\frac{\pi}{t}} e^{-\frac{(x-a)^2}{4t}} \sqrt{\frac{\pi}{t}} e^{-\frac{(y-b)^2}{4t}} dx\ dy \\
&= \int_{\mathbb{R}^2} f(x, y) \frac{1}{4\pi t} e^{-\frac{(x-a)^2+(y-b)^2}{4t}} dx\ dy
\end{aligned} \tag{2.1.5}$$

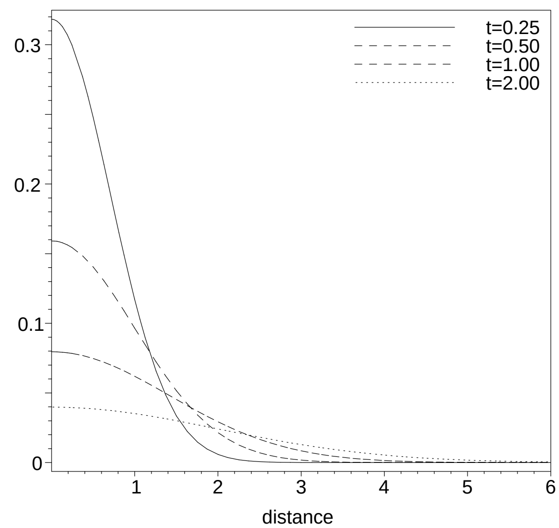

Figure 2.1.1: Graphs of the cartesian heat kernel

This means the heat kernel on $\mathbb{R}^2$ in cartesian coordinates is

$$K(x, y; a, b; t) = \frac{1}{4\pi t} e^{-\frac{(x-a)^2+(y-b)^2}{4t}}. \tag{2.1.6}$$

Figure 2.1.1 show graphs of the Cartesian heat kernel at various times t, with the horizontal axis showing the distance between the two points.

Remark 2.1.1 How does the kernel work in the solution of the heat equation on forms? If constructed as above it works in the following manner:

$$u(\mathbf{x}, t) = \int_M K(\mathbf{x}, \mathbf{y}, t) \wedge_{\mathbf{y}} *f(\mathbf{y}) = \langle K(\mathbf{x}, \mathbf{y}, t), f(\mathbf{y}) \rangle_g .$$

The subscript notation on the wedge indicates that it is the $\mathbf{y}$ variable that we are integrating with respect to. The function f is the initial condition for the heat equation. Since both the kernel and the initial conditions are of the same degree, the wedge product will guarantee a volume form for the integral. $\square$

The function K, being the heat kernel, satisfies the following:

1. K is symmetric in the space variables, in evaluation under integration. [7]

2. Since the Laplacian and the time derivative are invariant under isometries, g, we have $K(g\mathbf{x}, g\mathbf{y}, t) = K(\mathbf{x}, \mathbf{y}, t)$. This implies that $K(\mathbf{x}, \mathbf{y}, t)$ depends only on t and $d(\mathbf{x}, \mathbf{y})$, the distance between $\mathbf{x}$ and $\mathbf{y}$. $K(\mathbf{x}, \mathbf{y}, t)$ takes on values in $T^*_{\mathbf{x}}(M) \otimes T^*_{\mathbf{y}}(M)$. [15]

3. For $\mathbf{y_0} \in M$, $K(\mathbf{x}, \mathbf{y_0}, t)$ is a solution to the (inhomogenous) heat equation $(\Delta + \frac{\partial}{\partial t})K(\mathbf{x}, \mathbf{y_0}, t) = \delta(\mathbf{x}, \mathbf{y_0})\delta(t)$. This means, referring back to Section 1.6, that if the heat kernel on k-forms is known, we get the $n-k$-form heat kernel back as follows:

$$K_{n-k} = *_{\mathbf{x}} *_{\mathbf{y}} K_k,$$

since $\Delta *_{\mathbf{x}} *_{\mathbf{y}} = *_{\mathbf{x}} *_{\mathbf{y}} \Delta$.

2.2 The Heat Equation on Forms

For this section we will be focusing on the Euclidean plane with both the Cartesian and polar metrics.

Since the Laplacian preserves the degree of the forms it acts on, the heat equation on $\bigwedge^*(M)$ becomes $n+1$ separate heat equations, one for each $\bigwedge^k(M)$. We will show, for 0-forms, this Laplacian is exactly what we expect in $\mathbb{R}^2$.

Let us start with calculating the three different Laplacians for $\mathbb{R}^2$. In this case the Riemannian metric is the identity matrix, so $g = \det g_{ij} = 1$, thus the Hodge star and d^* are greatly simplified.

Since d^* takes 0-forms to 0, the Laplacian on 0-forms is just d^*d, so we need only calculate

$$\begin{aligned}
d^*d\omega &= (-1)^{2(1)+2+1} * d * (d\omega) \\
&= - * d * (\omega_{x_1}dx_1 + \omega_{x_2}dx_2) \\
&= - * d(\omega_{x_1}dx_2 - \omega_{x_2}dx_1) \\
&= - * [(\omega_{x_1x_1} + \omega_{x_2x_2})dx_1 \wedge dx_2] \\
&= -\omega_{x_1x_1} - \omega_{x_2x_2} \qquad (2.2.1)
\end{aligned}$$

The negative sign ensures that the spectrum is non-negative. So we can see that for 0-forms on $\mathbb{R}^2$, the Laplacian, $\Delta^{(0)}$, is exactly what we expected. The next easiest case is for 2-forms, of the form $\omega dx_1 \wedge dx_2$. In this instance, $d^*d(\omega dx_1 \wedge dx_2) = 0$.

$$\begin{aligned}
dd^*(\omega dx_1 \wedge dx_2) &= d(-1)^{2(2)+2+1} * d * (\omega dx_1 \wedge dx_2) \\
&= -d * d\omega \\
&= -d * (\omega_{x_1}dx_1 + \omega_{x_2}dx_2) \\
&= -d(\omega_{x_1}dx_2 - \omega_{x_2}dx_1) \\
&= -(\omega_{x_1x_1} + \omega_{x_2x_2})dx_1 \wedge dx_2 \qquad (2.2.2)
\end{aligned}$$

Hence, $\Delta^{(2)}(\omega dx_1 \wedge dx_2) = (\Delta^{(0)}\omega)dx_1 \wedge dx_2$. The final case will be dealt with in two parts, one for each of the summands in the Laplacian. Let $\omega dx_1 + \nu dx_2$ be a

1-form.

$$\begin{aligned}
dd^*(\omega dx_1 + \nu dx_2) &= d(-1)^{2(1)+2+1} * d * (\omega dx_1 + \nu dx_2) \\
&= -d * d(\omega dx_2 - \nu dx_1) \\
&= -d * [(\omega_{x_1} + \nu_{x_2}) dx_1 \wedge dx_2] \\
&= -d(\omega_{x_1} + \nu_{x_2}) \\
&= -(\omega_{x_1 x_1} + \nu_{x_2 x_1}) dx_1 - (\omega_{x_1 x_2} + \nu_{x_2 x_2}) dx_2 \qquad (2.2.3)
\end{aligned}$$

$$\begin{aligned}
d^* d(\omega dx_1 + \nu dx_2) &= (-1)^{2(2)+2+1} * d * d(\omega dx_1 + \nu dx_2) \\
&= - * d * [(-\omega_{x_2} + \nu_{x_1}) dx_1 \wedge dx_2] \\
&= - * d(\nu_{x_1} - \omega_{x_2}) \\
&= - * [(\nu_{x_1 x_1} - \omega_{x_2 x_1}) dx_1 + (\nu_{x_1 x_2} - \omega_{x_2 x_2}) dx_2] \\
&= -(\omega_{x_2 x_2} - \nu_{x_1 x_2}) dx_1 - (\nu_{x_1 x_1} - \omega_{x_2 x_1}) dx_2 \qquad (2.2.4)
\end{aligned}$$

Adding these two results we get

$$\Delta^{(1)}(\omega dx_1 + \nu dx_2) = (\Delta^{(0)}\omega) dx_1 + (\Delta^{(0)}\nu) dx_2.$$

Thus, in the Euclidean case, and only in this case, the Laplacian on arbitrary forms can be expressed as the 0-form Laplacian on the coefficients. Therefore, as we can solve the 0-form case, the solutions for the others follow.

The change of variables from Cartesian to polar coordinates changes the metric as well. For polar coordinates,

$$[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}.$$

This means the Laplacians are as follows:

$$\Delta^{(0)}u = -u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta} \tag{2.2.5}$$

$$\begin{aligned}\Delta^{(1)}(udr + vd\theta) &= -\left(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u + \frac{1}{r^2}u_{\theta\theta} - \frac{2}{r^3}v_\theta\right)dr \\ &\quad -\left(v_{rr} - \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} + \frac{2}{r}u_\theta\right)d\theta\end{aligned} \tag{2.2.6}$$

$$\Delta^{(2)}udr \wedge d\theta = -\left(u_{rr} - \frac{1}{r}u_r + \frac{1}{r^2}u + \frac{1}{r^2}u_{\theta\theta}\right)dr \wedge d\theta \tag{2.2.7}$$

Note that Δ_i, $i = 1, 2$ does not simplify to Δ_0, as it did in the Cartesian case. This is because the metric is dependent on the variable r, which gets introduced into the equation via the co-derivative, d^*.

2.3 The Heat Kernel for Forms: Examples

The heat kernel is a two-point form, meaning that

$$K_k : M \times M \times \mathbb{R}^+ \to \bigwedge^k T^*_{\mathbf{x}}M \bigotimes \bigwedge^k T^*_{\mathbf{y}}M$$

where K_k is called the k-form heat kernel. The heat kernel is a Green's function for the heat equation

$$(\Delta + \partial_t)\omega(\mathbf{x}, t) = 0,$$

which means, as mentioned in Remark 2.1.1, that

$$\omega(\mathbf{x}, t) = \langle K(\mathbf{x}, \mathbf{y}, t), f(\mathbf{y})\rangle_g$$

where $f(\mathbf{y})$ is the initial condition. The heat kernel is symmetric in the space variables, and by definition, it satisfies the heat equation with the Dirac delta functions as initial conditions. It can be shown that $*_{\mathbf{x}} *_{\mathbf{y}} K_k = K_{n-k}$, and so, in the case of surfaces, we know that $K_1 = *_{\mathbf{x}} *_{\mathbf{y}} K_1$ and $K_2 = *_{\mathbf{x}} *_{\mathbf{y}} K_0$.

Remark 2.3.1 The 0-form heat kernel for $\mathbb{R}^2$ is the classical function heat kernel which we recall from equation (2.1.6), and also from many other sources, such as [5], is

$$K_0(\mathbf{x},\mathbf{y},t) = \frac{1}{4\pi t} e^{-\frac{1}{4t}((x_1-y_1)^2+(x_2-y_2)^2)} \, 1_x \otimes 1_y.$$

The 1-form heat kernel for $\mathbb{R}^2$ can be determined by inspection of the differential equations.

$$K_1(\mathbf{x},\mathbf{y},t) = \frac{1}{4\pi t} e^{-\frac{1}{4t}((x_1-y_1)^2+(x_2-y_2)^2)} (dx_1 \otimes dy_1 + dx_2 \otimes dy_2).$$

The 2-form heat kernel for $\mathbb{R}^2$ is derived via the Hodge isomorphism.

$$K_2(\mathbf{x},\mathbf{y},t) = \frac{1}{4\pi t} e^{-\frac{1}{4t}((x_1-y_1)^2+(x_2-y_2)^2)} \, dx_1 \wedge dx_2 \otimes dy_1 \wedge dy_2.$$

$\square$

In the case of the 1-form heat kernel, in Eucildean polar coordinates, we have the option of merely doing a change of coordinates, using the known expression of the Cartesian 1-form heat kernel. It is more instructive, however, if we follow a similar approach using eigenforms.

Using integral transforms, we solve for the 0-form and 2-form heat kernel in the Euclidean polar case. To do this, we fix one of the points, (ρ, ϕ) at the origin and observe the heat kernel is rotationally symmetric, that is, it does not depend on the angular variable. Thus the angular derivatives disappear from the heat equation. We will denote the radial heat equation:

$$(\Delta_r^{(i)} + \partial_t)\omega(r,t) = f(r)\delta(t)$$

where ω and f are i-forms and f is the initial condition, at $t = 0$. We will work out in detail both the 0-form and the 2-form case.

Since the heat kernel takes as an argument two points on the surface, we fix one point, $\mathbf{y}$, at the orgin so as to avoid complications in the calculation. We will later remove this restriction of the heat kernel by translating the fixed point to an arbitary point.

Because the heat kernel is invariant under isometry and a rotation about a point is an isometry for $\mathbb{R}^2$, the heat kernel is rotationally symmetric, and therefore independent of θ, with one point fixed at the origin, and since the corresponding initial conditions are also rotationally symmetric, we will ignore the θ derivatives in the Laplacians when determining the heat kernel.

Example 2.3.2 Let us consider the plane, $\mathbb{R}^2$, in polar coordinates. We will solve for the heat kernel using integral transforms, more precisely using an eigenfunction transform . Recall that for a self-adjoint operator, L, an eigenfunction is a solution to the equation $Lu = \lambda u$, where the real number λ is called the eigenvalue. The eigenfunction transform works via the global inner product: let E_λ be an eigenfunction of L with eigenvalue λ, and define $\widehat{u} = \langle u, E \rangle_g$. Then

$$\widehat{Lu} = \langle Lu, E \rangle_g = \langle u, LE \rangle_g = \langle u, \lambda E \rangle_g = \lambda \widehat{u}.$$

Thus the transform removes the operator L.

For the 0-form case, the radial Laplacian is

$$\Delta_r^0 = -\partial_{rr} - \frac{1}{r}\partial_r$$

which gives the eigenfunction equation:

$$(-\partial_{rr} - \frac{1}{r}\partial_r)E(r) = \lambda^2 E(r).$$

The bounded solutions of this equation are Bessel functions,

$$E_{\lambda^2}^0(r) = J_0(\lambda r),$$

where we use the subscript to denote the eigenvalue, and the superscript indicates that this is for the 0-form equation.

We note that the spectrum of the Laplacian is always positive, since if E is an eigenfunction with eigenvalue μ, then

$$\mu \langle E, E\rangle_g = \langle \Delta E, E\rangle_g = \langle dE, dE\rangle_g + \langle d^* E, d^* E\rangle_g \geq 0.$$

It can be shown that for the Euclidean polar 0-form case, the spectrum of the Laplacian is $[0, \infty)$.

We define the eigenfunction transform of $f(r)$ to be

$$\widehat{f}(\lambda) := \left\langle f(r), E^0_{\lambda^2}(r)\right\rangle_g = \int_{\mathbb{R}^2} f(r) \wedge *E^0_{\lambda^2}(r).$$

After performing the Hodge star operation, and integrating with respect to θ, we find

$$\widehat{f}(\lambda) = 2\pi \int_0^\infty f(r) J_0(\lambda r) r dr.$$

This is very close to the Hankel transform of order 0, and can be re-written as

$$\widehat{f}(\lambda) = \frac{2\pi}{\lambda^{1/2}} \int_0^\infty r^{1/2} f(r) J_0(\lambda r)(\lambda r)^{1/2} dr.$$

This matches the form of the Hankel transform given in [20], but it is the transform of $r^{1/2} f(r)$, not $f(r)$.

If we apply the eigenfunction transform to the radial heat equation we get

$$(\lambda^2 + \partial_t)\widehat{\omega}(\lambda, t) = \widehat{f}(\lambda)\delta(t)$$

since the Laplacian is self-adjoint in the given inner product. By applying the Laplace transform to the time variable, we arrive at

$$(\lambda^2 + s)\widetilde{\widehat{\omega}}(\lambda, s) = \widehat{f}(\lambda)$$

or

$$\widetilde{\widehat{\omega}}(\lambda, s) = \frac{\widehat{f}(\lambda)}{\lambda^2 + s}.$$

In this we assumed that $\widehat{\omega}(\lambda, 0) = 0$ in the Laplace transform, because the initial condition was already accounted for on the right-hand side of the equation.

Using a table of Laplace transforms, [42], we find that

$$\widehat{\omega}(\lambda, t) = \widehat{f}(\lambda)e^{-\lambda^2 t}.$$

Thus we have

$$\frac{2\pi}{\lambda^{1/2}} \int_0^\infty r^{1/2}\omega(r,t)J_0(\lambda r)(\lambda r)^{1/2}dr = \frac{2\pi e^{-\lambda^2 t}}{\lambda^{1/2}} \int_0^\infty r^{1/2} f(r)J_0(\lambda r)(\lambda r)^{1/2}dr$$

or

$$\int_0^\infty r^{1/2}\omega(r,t)J_0(\lambda r)(\lambda r)^{1/2}dr = e^{-\lambda^2 t} \int_0^\infty r^{1/2} f(r)J_0(\lambda r)(\lambda r)^{1/2}dr.$$

We know that the Hankel transform is self-inverse, [20], so

$$\rho^{1/2}\omega(\rho, t) = \int_0^\infty e^{\lambda^2 t} \int_0^\infty r^{1/2} f(r)J_0(\lambda r)(\lambda r)^{1/2}dr\, J_0(\rho\lambda)(\rho\lambda)^{1/2}d\lambda.$$

With some manipulation, we have

$$\omega(\rho, t) = \int_0^\infty r f(r) \int_0^\infty \lambda e^{\lambda^2 t} J_0(\lambda r)J_0(\rho\lambda)d\lambda dr.$$

Next, we use the translation invariance of the heat kernel to simplify the inner integral by setting $(\rho, \phi) = (0, 0)$. Since $J_0(0) = 1$, we have

$$\int_0^\infty \lambda e^{-\lambda^2 t} J_0(\lambda r)d\lambda = r^{-\frac{1}{2}} \int_0^\infty \lambda^{\frac{1}{2}} e^{-\lambda^2 t} J_0(\lambda r)(\lambda r)^{\frac{1}{2}} d\lambda$$

which is, in turn, by [20, eq. 8.6.10, $\nu = 0$],

$$\frac{r^{\frac{1}{2}}}{2t} e^{-\frac{r^2}{4t}}.$$

Thus we have

$$\omega(0,t) = \int_0^\infty r f(r) \frac{1}{2t} e^{-\frac{r^2}{4t}}\, dr.$$

If we use this result to solve the heat equation, we observe that in fact we have found a Green's function

$$K(r,\theta;0,0;t) = \frac{1}{4\pi t} e^{-r^2/4t}$$

and that

$$\omega(0,t) = \langle f(r), K(r,\theta;0,0;t)\rangle_g = \int_{\mathbb{R}^2} f(r) r K(r,\theta;0,0;t) dr \wedge d\theta.$$

The extra $1/2\pi$ comes from introducing the integration over θ.

All that remains is translating $(0,0)$ to (ρ,ϕ). We note that the r in the heat kernel represents the distance from (r,θ) to $(0,0)$. If we replace r by the distance from (r,θ) to (ρ,ϕ), which is $\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\phi)}$, we arrive at the well-known heat kernel

$$K(r,\theta;\rho,\phi;t) = \frac{1}{4\pi t} e^{-\dfrac{r^2+\rho^2-2r\rho\cos(\theta-\phi)}{4t}}. \tag{2.3.1}$$

For the 2-form case, the radial Laplacian is

$$\Delta_r^2 = -\partial_{rr} + \frac{1}{r}\partial_r - \frac{1}{r^2}$$

or, if we consider normalized forms,

$$\Delta_r^2 \omega \overline{dr \wedge d\theta} = -(\partial_{rr}\omega + \frac{1}{r}\partial_r \omega)\overline{dr \wedge d\theta}$$

and the eigenform with eigenvalue λ^2 is

$$E_{\lambda^2}^2(r) = rJ_0(\lambda r) dr \wedge d\theta = J_0(\lambda r)\overline{dr \wedge d\theta}.$$

We define the eigenform transform of $f(r)\overline{dr \wedge d\theta}$ to be

$$\widehat{f}(\lambda) := \left\langle f(r)\overline{dr \wedge d\theta}, E^2_{\lambda^2}(r)\right\rangle_g = \int_0^\infty \int_0^{2\pi} f(r)\overline{dr \wedge d\theta} \wedge *E^2_{\lambda^2}(r).$$

After performing the Hodge star operation, we find

$$\widehat{f}(\lambda) = 2\pi \int_0^\infty r f(r) J_0(\lambda r) dr.$$

At this point, we can plainly see the process follows exactly as in the 0-form case, considering the normalized case. †

We now turn our attention to the 1-form case. As before, we will assume initial conditions which are independent of the angular variable. This implies the solution of the heat equation will also be independent of the angular variable. Thus, the Laplacian applied to a 1-form, $u(r,t)\overline{dr} + v(r,t)\overline{d\theta}$, is

$$\Delta^{(1)}(u\overline{dr} + v\overline{d\theta}) = -(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u)\overline{dr} - (v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v)\overline{d\theta}$$

It can be checked that an eigenform, with eigenvalue λ^2, of the Laplacian is $J_1(\lambda r)(\overline{dr} + \overline{d\theta})$. We will use this with the global inner product as an integral transform.

Let us recall the exact statement of the problem we are trying to solve:

$$\left\{ \begin{array}{c} (\Delta^{(1)} + \partial_t)\omega(r,t) = 0 \\ \omega(r,0) = a(r)\overline{dr} + b(r)\overline{d\theta} \end{array} \right\} \tag{2.3.2}$$

where $\omega(r,t) = u(r,t)\overline{dr} + v(r,t)\overline{d\theta}$, and as we stated previously, since the initial conditions are independent of θ, so is ω. Since the Laplacian does not mix the $\overline{dr}$ and $\overline{d\theta}$ terms, and the heat equation is linear, we will consider the $\overline{dr}$ and $\overline{d\theta}$ terms separately.

As before, we will write

$$\widehat{u}(\lambda) := \left\langle u(r)\overline{dr}, J_1(\lambda r)(\overline{dr} + \overline{d\theta})\right\rangle_g = \int_0^{2\pi}\int_0^{\infty} u(r)\overline{dr} \wedge *J_1(\lambda r)(\overline{dr} + \overline{d\theta}) \quad (2.3.3)$$

This simplifies to

$$\widehat{u}(\lambda) = 2\pi \int_0^{\infty} ru(r)J_1(\lambda r)dr \quad (2.3.4)$$

since integration is performed against the unnormalized forms.

If we apply this transform to equation (2.3.2), restricting to the $\overline{dr}$ portion, we have

$$(\lambda^2 + \partial_t)\widehat{u}(\lambda, t) = 0.$$

Then applying the Laplace transform to the time variable, and rearranging, we arrive at

$$\tilde{\widehat{u}}(\lambda, \tau) = \frac{\widehat{a}(\lambda)}{\lambda^2 + \tau}.$$

Applying the inversion of the Laplace transform yields

$$\widehat{u}(\lambda, t) = \widehat{a}(\lambda)e^{-\lambda^2 t} = 2\pi e^{-\lambda^2 t}\int_0^{\infty} ra(r)J_1(\lambda r)dr. \quad (2.3.5)$$

At this point, further simplification is not productive. Since this is the polar coordinate Euclidean plane, we have the advantage of knowing what the result should be. So we will now approach the problem from the viewpoint of already knowing the heat kernel in cartesian coordinates, with the goal of arriving at (2.3.5). Recall from Remark 2.3.1 that

$$K^1_{cartesian}(\mathbf{x}, \mathbf{y}, t) = \frac{1}{4\pi t}e^{-\frac{(x^1-y^1)^2+(x^2-y^2)^2}{4t}}\left(dx^1 \otimes dy^1 + dx^2 \otimes dy^2\right).$$

If we convert this expression to polar coordinates, using

$$x^1 = r\cos\theta, \quad x^2 = r\sin\theta$$

$$y^1 = \rho\cos\phi, \quad y^2 = \rho\sin\phi$$

we arrive at, with $\mathbf{r} = (r, \theta)$ and $\mathbf{s} = (\rho, \phi)$,

$$K^1_{polar}(\mathbf{r}, \mathbf{s}, t) = \frac{1}{4\pi t} e^{-\frac{r^2+\rho^2-2r\rho\cos(\theta-\phi)}{4t}} \Omega \tag{2.3.6}$$

where

$$\begin{aligned}
\Omega &= \cos(\theta-\phi)\overline{dr}\otimes\overline{d\rho} + \sin(\theta-\phi)\overline{dr}\otimes\overline{d\phi} \\
&\quad - \sin(\theta-\phi)\overline{d\theta}\otimes\overline{d\rho} + \cos(\theta-\phi)\overline{d\theta}\otimes\overline{d\phi} \\
&= d_{\mathbf{r}} d_{\mathbf{s}}(r\rho\cos(\theta-\phi))
\end{aligned} \tag{2.3.7}$$

The solution of the heat equation (2.3.2) can be written as

$$\omega(\mathbf{r}, t) = \int_{\mathbb{R}^2} K^1_{polar}(\mathbf{r}, \mathbf{s}, t) \wedge *\omega(\mathbf{s}, 0).$$

If we substitute in the expressions for K^1_{polar} and $\omega(\mathbf{r}, 0)$ taken from (2.3.2) we see

$$\omega(\mathbf{r}, t) = \frac{e^{-\frac{r^2}{4t}}}{4\pi t} \int_0^\infty e^{-\frac{\rho^2}{4t}} \int_0^{2\pi} e^{\frac{r\rho\cos(\theta-\phi)}{2t}} \alpha \otimes \rho d\phi d\rho$$

with

$$\alpha = [a(\rho)\cos(\theta-\phi) + b(\rho)\sin(\theta-\phi)]\,\overline{dr} + [b(\rho)\cos(\theta-\phi) - a(\rho)\sin(\theta-\phi)]\,\overline{d\theta}.$$

since $\overline{d\rho}\wedge\overline{d\phi} = \rho d\rho\wedge d\phi$ and when integrating, we can switch order without changing sign (that is, after it is in standard form, we associate $d\rho \wedge d\phi$ with $d\rho d\phi$).

To evaluate the inner integral, consider

$$\int_0^{2\pi} e^{y\cos x} \left\{ \begin{array}{c} \cos x \\ \sin x \end{array} \right\} dx = \left\{ \begin{array}{l} 2\pi I_1(y) \\ 0 \end{array} \right. .$$

This allows us to remove the $\sin(\theta - \phi)$ terms and replace the $\cos(\theta - \phi)$ terms with a modified Bessel function of the first kind. We notice also that this removes the θ dependence in the integral allowing us to write

$$\omega(r, t) = \frac{e^{-\frac{r^2}{4t}}}{2t} \int_0^\infty \rho e^{-\frac{\rho^2}{4t}} I_1\left(\frac{r\rho}{2t}\right) \left[a(\rho)\overline{dr} + b(\rho)\overline{d\theta}\right] d\rho.$$

At this point we separate the integral into the $\overline{dr}$ and the $\overline{d\theta}$ terms and focus on the $\overline{dr}$ portion. This corresponds to

$$u(r,t)\overline{dr} = \left(\frac{e^{-\frac{r^2}{4t}}}{2t} \int_0^\infty \rho e^{-\frac{\rho^2}{4t}} I_1\left(\frac{r\rho}{2t}\right) a(\rho) d\rho \right) \overline{dr}.$$

Using equation (2.3.4) we have

$$\widehat{u}(\lambda,t) = 2\pi \int_0^\infty r \frac{e^{-\frac{r^2}{4t}}}{2t} \int_0^\infty \rho e^{-\frac{\rho^2}{4t}} I_1\left(\frac{r\rho}{2t}\right) a(\rho) d\rho J_1(\lambda r) dr$$

Rearranging this we find

$$\widehat{u}(\lambda,t) = \frac{\pi}{t\sqrt{\lambda}} \int_0^\infty \rho e^{-\frac{\rho^2}{4t}} a(\rho) \int_0^\infty \sqrt{r} e^{-\frac{r^2}{4t}} I_1\left(\frac{r\rho}{2t}\right) J_1(\lambda r)(r\lambda)^{\frac{1}{2}} dr\, d\rho.$$

Evaluating the inner integral with respect to r, we see that the equation reduces to

$$\widehat{u}(\lambda,t) = 2\pi e^{-\lambda^2 t} \int_0^\infty \rho a(\rho) J_1(\lambda\rho) d\rho, \tag{2.3.8}$$

which, except for the variable of integration, is identical to equation (2.3.5). Since the steps are reversible, we have shown equality of both sides. The same calculations can be carried through for the $\overline{d\theta}$ term.

Chapter 3
The Hyperbolic Plane

3.1 Models of the Hyperbolic Plane

There are two well-known models of the hyperbolic plane. The first is the Poincaré disk [29, 30, 38, 45], which we will denote P. The Poincaré disk is the open unit disk in the Euclidean plane, but geodesics of the hyperbolic plane became arcs of circles which meet the perimeter of the disk at right angles. The second model is the upper half-plane model, [3, 11, 38, 45], which we will denote U. This model, as the name suggests, is the upper half-plane in the Euclidean plane, with the horizontal axis removed. Geodesics in this model are arcs of circles which meet the horizontal axis at right angles.

There is another model, found in [10, 38], which is a surface in the Lorentzian space, $\mathbb{R}^{1,2}$, also known as 3-dimensional Minkowski space-time. The metric in $\mathbb{R}^{1,2}$ is

$$ds^2 = -dx^2 + dy^2 + dz^2. \tag{3.1.1}$$

The surface in this space we will consider for our model is

$$H^2 := \left\{ \mathbf{x} \in \mathbb{R}^{1,2} | -x^2 + y^2 + z^2 = -1, \;\; x > 0 \right\}.$$

This can be pictured as the upper sheet of a two-sheet hyperboloid, as seen in Figure

3.1.1. This model of the hyperbolic plane can be parameterized by

$$\begin{aligned} x &= \cosh\eta \\ y &= \sinh\eta\cos\theta \\ z &= \sinh\eta\sin\theta \end{aligned} \qquad (3.1.2)$$

Substituting this parameterization into equation (3.1.1) we find that the metric on H^2 is

$$ds^2 = d\eta^2 + \sinh^2\eta d\theta^2, \qquad (3.1.3)$$

or equivalently,

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2\eta \end{bmatrix}. \qquad (3.1.4)$$

Using the Theorema Egregium of Gauss, [32], we can calculate the curvature of this space from the metric and its derivatives, showing that this model of the hyperbolic plane has constant negative curvature, in this case with the given metric, -1. The other two models, P and U, also have constant negative curvature, and with suitable scaling we can have curvature -1. If we compare the hyperboloid model of the hyperbolic plane to the unit sphere, a surface of constant postive curvature 1 with metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$, we could imagine that H^2 is like a sphere of imaginary radius.

Using change of coordinates, we can move from one model of the hyperbolic plane to another, since they are all the same smooth manifold, much like going from Cartesian to polar coordinates in the Euclidean plane. The mapping from H^2 to P can be visualized as is Figure 3.1.1, by taking the line joining the vertex of the lower sheet of the hyperboloid to a point in H^2 and finding the intersection with the plane $x = 0$. [1]

[1]This process resembles the projection of the Riemann sphere onto the complex plane.

Model	Poincaré Disk (P)	Upper Half-plane (U)	Hyperboloid (H^2)
Coordinates	$x^2 + y^2 < 1$	$X \in \mathbb{R}$ $Y > 0$	$\eta \geq 0$ $\theta \in [0, 2\pi)$
ds^2	$\dfrac{4}{(1 - x^2 - y^2)^2} \left(dx^2 + dy^2\right)$	$\dfrac{1}{Y^2} dX^2 + \dfrac{1}{Y^2} dY^2$	$d\eta^2 + \sinh^2 \eta d\theta^2$
$\Delta^{(0)}$	$-\dfrac{(1 - x^2 - y^2)^2}{4} (\partial_{xx} + \partial_{yy})$	$-Y^2(\partial_{XX} + \partial_{YY})$	$-\partial_{\eta\eta} - \coth \eta \partial_\eta - \mathrm{csch}^2 \eta \partial_{\theta\theta}$
cosh of $d_M(\mathbf{x}, \mathbf{y})$	$\dfrac{(1 + x_1^2 + x_2^2)(1 + y_1^2 + y_2^2) - 4(x_1 y_1 + x_2 y_2)}{(1 - x_1^2 - x_2^2)(1 - y_1^2 - y_2^2)}$	$\dfrac{(X_1 - Y_1)^2 + X_2^2 + Y_2^2}{2 X_2 Y_2}$	$\cosh \eta \cosh \rho - \sinh \eta \sinh \rho \cos(\theta - \phi)$

Table 3.1.1: Models of H^2

Relationship between models $H^2 : (\eta, \theta)$, $U : (X, Y)$, and $P : (x, y)$
Upper half-plane (U) and Poincaré disc (P) $X = \dfrac{2x}{x^2 + (y-1)^2} \qquad Y = \dfrac{1 - x^2 - y^2}{x^2 + (y-1)^2}$ $x = \dfrac{2X}{X^2 + (Y+1)^2} \qquad y = \dfrac{X^2 + Y^2 - 1}{X^2 + (Y+1)^2}$
Hyperboloid (H^2), upper half-plane (U) and Poincaré disc (P) $\tanh^2(\dfrac{\eta}{2}) = \dfrac{X^2 + (Y-1)^2}{X^2 + (Y+1)^2} = x^2 + y^2$ $\tan\theta = \dfrac{X^2 + Y^2 - 1}{2X} = \dfrac{y}{x}$

Table 3.1.2: Relationships between models of the hyperbolic plane

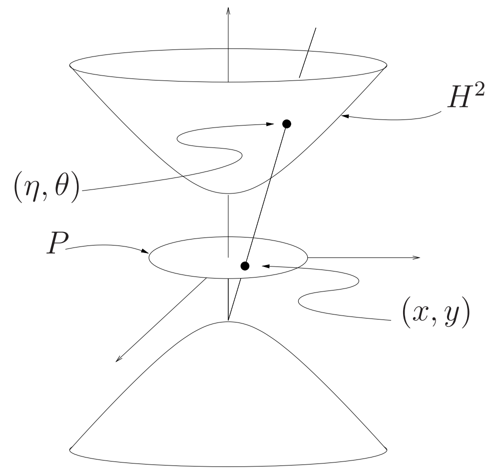

Figure 3.1.1: Mapping H^2 to P

For the purposes of this section, we will be concentrating on the hyperboloid model of H^2. We choose this model because it has a special point, the origin, which makes the problem of solving for the heat kernel somewhat simpler by introducing rotational symmetry. The Poincaré disk can also be given a polar coorinate metric via the usual coordinate transformation. In [9, 11] solutions of the 0-form heat kernel in the hyperboloid model are constructed, so we choose this metric over the Poincaré disk. Using rotational symmetry, we can consider forms which depend only on the distance from the origin, thus simplifying the differential equations, as we have seen in Section 2.3.

3.2 The Heat Equation on H^2

Recall from Section 2.2, that the heat equation on differential forms is written,

$$(\Delta + \partial_t)\omega(\mathbf{x}, t) = 0 \tag{3.2.1}$$
$$\omega(\mathbf{x}, 0) = \nu(\mathbf{x}),$$

where ω is a differential form depending on a point $\mathbf{x} \in M$ and on time, t, with initial condition $\nu(\mathbf{x})$. As mentioned previously, $\Delta = (d + d^*)^2 = dd^* + d^*d$ maps k-forms to k-forms, and is a linear differential operator. Thus we do not need to consider mixed degree forms. This means, in our situation, the heat equation is actually three separate equations, one for 0-forms, which is the classical case, one for 2-forms, which is isomorphic to the 0-form case, and one for 1-forms, which yields two coupled differential equations.

Remark 3.2.1 Donnelly [14] gives an explicit formula for the Laplacian on H^{n+1}, with separation of angular and radial variables. Given the metric in H^{n+1} is of the form

$$ds^2 = dr^2 + (g(r))^2 d\omega^2$$

where $g(r) = \sinh r$, and $d\omega$ is the standard metric on S^n, we can write the Laplacian as

$$\begin{aligned}\Delta\phi &= g^{-2}\Delta_S\phi - g^{2p-n}\partial_r(g^{n-2p}\partial_r\phi_1) - \partial_r(g^{2p-n-2}\partial_r(g^{n-2p+2}\phi_2))dr \\ &\quad +2(-1)^p g^{-1}\partial_r g(d_S\phi_2 + g^{-2}d_S^*\phi_1 \wedge dr)\end{aligned} \tag{3.2.2}$$

where the p-form ϕ is written $\phi_1 + \phi_2 \wedge dr$, ϕ_2 is a $p-1$-form with support on S^n, and operators with subscript S act on S^n.

In our situation, with $n = 1$, we have

$$\begin{aligned}\Delta\phi &= g^{-2}\partial_{\theta\theta}\phi - g^{2p-1}\partial_r(g^{1-2p}\partial_r\phi_1) - \partial_r(g^{2p-3}\partial_r(g^{3-2p}\phi_2))dr \\ &\quad +2(-1)^p g^{-1}\partial_r g(d_S\phi_2 + g^{-2}d_S^*\phi_1 \wedge dr)\end{aligned} \tag{3.2.3}$$

□

For the case of the hyperbolic plane, we will use the hyperboloid model, H^2, with metric given by (3.1.4). The heat equation thus computed, is,

$$\begin{aligned}
0 &= (\Delta+\partial_t)\left(\mu+\omega\overline{d\eta}+\nu\overline{d\phi}+\tau\overline{d\eta\wedge d\phi}\right)\\
&= \left(-\mu_{\eta\eta}-\frac{\cosh\eta}{\sinh\eta}\mu_\eta-\frac{1}{\sinh^2\eta}\mu_{\phi\phi}+\mu_t\right)\\
&\quad+\left(-\omega_{\eta\eta}-\frac{\cosh\eta}{\sinh\eta}\omega_\eta+\frac{1}{\sinh^2\eta}\omega-\frac{1}{\sinh^2\eta}\omega_{\phi\phi}+2\frac{\cosh\eta}{\sinh\eta}\nu_\phi+\omega_t\right)\overline{d\eta}\\
&\quad+\left(-\nu_{\eta\eta}-\frac{\cosh\eta}{\sinh\eta}\nu_\eta+\frac{1}{\sinh^2\eta}\nu-\frac{1}{\sinh^2\eta}\nu_{\phi\phi}-2\frac{\cosh\eta}{\sinh\eta}\omega_\phi+\nu_t\right)\overline{d\phi}\\
&\quad+\left(-\tau_{\eta\eta}-\frac{\cosh\eta}{\sinh\eta}\tau_\eta-\frac{1}{\sinh^2\eta}\tau_{\phi\phi}+\tau_t\right)\overline{d\eta\wedge d\phi}. \qquad (3.2.4)
\end{aligned}$$

Notice that the $\overline{d\eta}$ and $\overline{d\phi}$ portion of the equation are coupled with a derivative in ϕ. If we consider the Laplacian applied only to a radially symmetric argument, then the equations decouple, with all terms involving a derivative in ϕ disappearing. For forms independent of ϕ, the heat equation is

$$\begin{aligned}
0 &= (\Delta+\partial_t)\left(\mu+\omega\overline{d\eta}+\nu\overline{d\phi}+\tau\overline{d\eta\wedge d\phi}\right)\\
&= \left(-\mu_{\eta\eta}-\frac{\cosh\eta}{\sinh\eta}\mu_\eta+\mu_t\right)\\
&\quad+\left(-\omega_{\eta\eta}-\frac{\cosh\eta}{\sinh\eta}\omega_\eta+\frac{1}{\sinh^2\eta}\omega+\omega_t\right)\overline{d\eta}\\
&\quad+\left(-\nu_{\eta\eta}-\frac{\cosh\eta}{\sinh\eta}\nu_\eta+\frac{1}{\sinh^2\eta}\nu+\nu_t\right)\overline{d\phi}\\
&\quad+\left(-\tau_{\eta\eta}-\frac{\cosh\eta}{\sinh\eta}\tau_\eta+\tau_t\right)\overline{d\eta\wedge d\phi}. \qquad (3.2.5)
\end{aligned}$$

Even though there is no dependence on ϕ in the forms, we still consider the $\overline{d\phi}$ and $\overline{d\eta\wedge d\phi}$ terms. With the restricted equation, the 0-form and the 2-form portion have the same structure, as do the $\overline{d\eta}$ and $\overline{d\phi}$ portions. For this reason, we will consider only the 0-form case and the $\overline{d\eta}$ portion of the 1-form equation, and then extend the results to the remainder of the terms.

3.3 The 0-form Heat Kernel

Following the method found in Chavel [9], we will derive the heat kernel for 0-forms. From (3.2.4) we see that the 0-form, or function, heat equation is

$$-\mu_{\eta\eta} - \frac{\cosh\eta}{\sinh\eta}\mu_\eta - \frac{1}{\sinh^2\eta}\mu_{\phi\phi} + \mu_t = 0,$$

with

$$\mu(\eta, \phi, 0) = f(\eta, \phi)$$

as the initial condition.

Assuming that we have a radial function μ, that is, μ a function of η and t only, we can drop the ϕ term in the equation. Note that η in this coordinate system also represents the distance from (η, ϕ) to the origin. Next, to help us solve this heat equation we will find the eigenfunction for the Laplacian, assuming radial symmetry. Thus we are trying to solve

$$\Delta E = -(\partial_{\eta\eta} + \coth\eta\partial_\eta)E = \lambda E,$$

where E depends only on η and t. If we substitute $x = \cosh\eta$ and $\lambda = -v(v+1)$ the equation transforms to

$$(1-x^2)E_{xx} - 2xE_x + v(v+1)E = 0,$$

which is Legendre's equation of order v. If we solve for v in terms of λ we find

$$v = \frac{-1 \pm \sqrt{1-4\lambda}}{2}.$$

Substituting $i\rho = \sqrt{1/4 - \lambda}$ and taking the positive branch of the solution for v we have as a solution to the eigenfunction equation

$$E(\eta) = P_{-1/2+i\rho}(\cosh\eta) =: F_\rho(\eta).$$

Next we define an eigenfunction transform as in Section 2.3

$$\widehat{\alpha}(\rho) = \int_0^\infty \alpha(\eta) F_\rho(\eta) \sinh \eta d\eta,$$

which happens to be the Mehler-Fock transform. The inverse transform is

$$\alpha(\eta) = \int_0^\infty \widehat{\alpha}(\rho) F_\rho(\eta) \rho \tanh \pi\rho d\rho.$$

Applying this transform to the heat equation to a function with radial symmetry yields

$$\left(\frac{1}{4} + \rho^2\right) \widehat{\alpha} + \widehat{\alpha}_t = 0, \tag{3.3.1}$$

which is an ordinary differential equation whose solution is $\widehat{\alpha} = \widehat{f}(\rho) e^{-(1/4+\rho^2)t}$, where $f(\eta)$ is the initial condition for the heat equation.

Now, the solution thus far resembles the product of two transformed function. It would then be reasonable to consider a convolution of functions as its inverse transform. Let us define the convolution of two functions as

$$(\xi * \psi)(\mathbf{z}) = \int_{H^2} \xi(\mathbf{w}) \psi(g_{\mathbf{w}}^{-1} \mathbf{z}) dV(\mathbf{w}).$$

In order to do this, we change coordinates from the hyperboloid model to the Poincaré disc model. This change of coordinates allows us to write $\mathbf{z} = re^{i\phi}$ with $r = \tanh(\eta/2)$ and $\mathbf{w} = Re^{i\sigma}$ with $R = \tanh(\eta_1/2)$. We should note that

$$g_{\mathbf{w}} \mathbf{z} = k_\sigma T_R \mathbf{z} = e^{i\sigma} \frac{\mathbf{z} + R}{1 + R\mathbf{z}}. \tag{3.3.2}$$

This operator $g_{\mathbf{w}}$ is an isometry on the Poincaré disc. We say that ξ is radial if $\xi(k_\tau \mathbf{z}) = \xi(\mathbf{z})$ for any rotation. It can be shown that if ξ has radial symmetry then $\xi * \psi$ does as well.

For ξ, ψ radial with respect to $\mathbf{z} = 0$, we have

$$\begin{aligned}
\widehat{\xi * \psi}(\rho) &= (2\pi)^{-1} \int_{H^2} F_\rho(\mathbf{z}) \int_{H^2} \xi(\mathbf{w}) \psi({g_\mathbf{w}}^{-1}\mathbf{z}) dV(\mathbf{w}) dV(\mathbf{z}) \\
&= (2\pi)^{-1} \int_{H^2} \xi(\mathbf{w}) \int_{H^2} F_\rho(\mathbf{z}) \psi({g_\mathbf{w}}^{-1}\mathbf{z}) dV(\mathbf{z}) dV(\mathbf{w}) \\
&= (2\pi)^{-1} \int_{H^2} \xi(\mathbf{w}) dV(\mathbf{w}) \int_{H^2} \psi(\mathbf{x}) F_\rho(g_\mathbf{w}\mathbf{x}) dV(\mathbf{x})
\end{aligned}$$

Since ψ has radial symmetry, we know

$$\int_{H^2} \psi(\mathbf{x}) F_\rho(g_\mathbf{w}\mathbf{x}) dV(\mathbf{x}) = \int_{H^2} \psi(k^{-1}\mathbf{x}) F_\rho(g_\mathbf{w}\mathbf{x}) dV(\mathbf{x}) = \int_{H^2} \psi(\mathbf{y}) F_\rho(g_\mathbf{w} k\mathbf{y}) dV(\mathbf{y}),$$

where k is any rotation. Then, using the Mean Value Theorem, we can write,

$$\int_{H^2} \psi(\mathbf{x}) F_\rho(g_\mathbf{w}\mathbf{x}) dV(\mathbf{x}) = \int_{H^2} \psi(\mathbf{y}) (2\pi)^{-1} \int_0^{2\pi} F_\rho(g_\mathbf{w} k_\tau \mathbf{y}) d\tau dV(\mathbf{y}).$$

Using [9], we know that

$$(2\pi)^{-1} \int_0^{2\pi} F_\rho(g_\mathbf{w} k_\tau \mathbf{y}) d\tau = F_\rho(\mathbf{w}) F_\rho(\mathbf{y}).$$

Thus

$$\widehat{\xi * \psi}(\rho) = 2\pi \widehat{\xi}(\rho) \widehat{\psi}(\rho).$$

In our case, where $\xi = f$ and $\widehat{\psi} = e^{-(1/4+\rho^2)t}$, we need to find the inverse transform of $e^{-(1/4+\rho^2)t}$, for which we simply use the inversion formula to get

$$\psi(\eta) = \int_0^\infty F_\rho(\eta) \rho e^{-(1/4+\rho^2)t} \tanh \pi\rho d\rho.$$

Thus we have

$$\alpha = (2\pi)^{-1} \int_{H^2} f(\mathbf{w}) dV(\mathbf{w}) \int_0^\infty F_\rho({g_\mathbf{w}}^{-1}\eta) \rho e^{-(1/4+\rho^2)t} \tanh \pi\rho d\rho.$$

From this it is clear to see that the heat kernel is

$$K_0(\mathbf{z}, \mathbf{w}, t) = (2\pi)^{-1} \int_0^\infty F_\rho({g_\mathbf{w}}^{-1}\mathbf{z}) \rho e^{-(1/4+\rho^2)t} \tanh \pi\rho d\rho,$$

with $\mathbf{z} = re^{i\phi}$. We should also note that $\cosh g_{\mathbf{w}}{}^{-1}\mathbf{z} = \cosh(d_{H^2}(\mathbf{z},\mathbf{w}))$, where $d_{H^2}(\mathbf{z},\mathbf{w})$ is the hyperbolic distance from $\mathbf{z}$ to $\mathbf{w}$.

Therefore, writing it out in full, the heat kernel for functions on H^2 is found to be

$$K_0^{(H^2)}(\mathbf{z},\mathbf{w},t) = (2\pi)^{-1}\int_0^\infty P_{-1/2+i\rho}(\cosh d_{H^2}(\mathbf{z},\mathbf{w}))\rho e^{-(1/4+\rho^2)t}\tanh \pi\rho d\rho. \quad (3.3.3)$$

After some simplification this agrees with the heat kernel given by McKean [35] and Davies [11]:

$$K_0^{(H^2)}(d_{H^2}(\mathbf{z},\mathbf{w}),t) = 2^{\frac{1}{2}}\,(4\pi t)^{-\frac{3}{2}}\,e^{-\frac{t}{4}}\int_{d_{H^2}(\mathbf{z},\mathbf{w})}^\infty \frac{se^{-\frac{s^2}{4t}}}{(\cosh s - \cosh d_{H^2}(\mathbf{z},\mathbf{w}))^{\frac{1}{2}}}ds.$$

In Figure 3.3.1, we show graphs of the integrand of the heat kernel, with the horizontal axis representing the distance between the two points $\mathbf{x},\mathbf{y}$. It is clear that these functions decay rapidly as both the distance and the eigenvalue parameter, ρ, increase. Using this information, we generate a numerical approximation of the heat kernel by limiting ρ to the interval $[0,5]$. The results, for various times t, are shown in Figure 3.3.2. These graphs are very similar to those of the heat kernel for $\mathbb{R}^2$ given in Figure 2.1.1.

3.4 Eigenform Method

In this section we will derive the 1-form heat kernel on H^2 using the hyperboloid metric. We will assume a radially symmetric 1-form, and as before, we will concentrate on the $\overline{d\eta}$ portion of the differential equation. This means we are looking at the problem

$$\left\{ \begin{array}{c} (\Delta + \partial_t)\,\omega(\eta,t)\overline{d\eta} = 0 \\ \omega(\eta,0)\overline{d\eta} = f(\eta)\overline{d\eta} \end{array} \right\}. \quad (3.4.1)$$

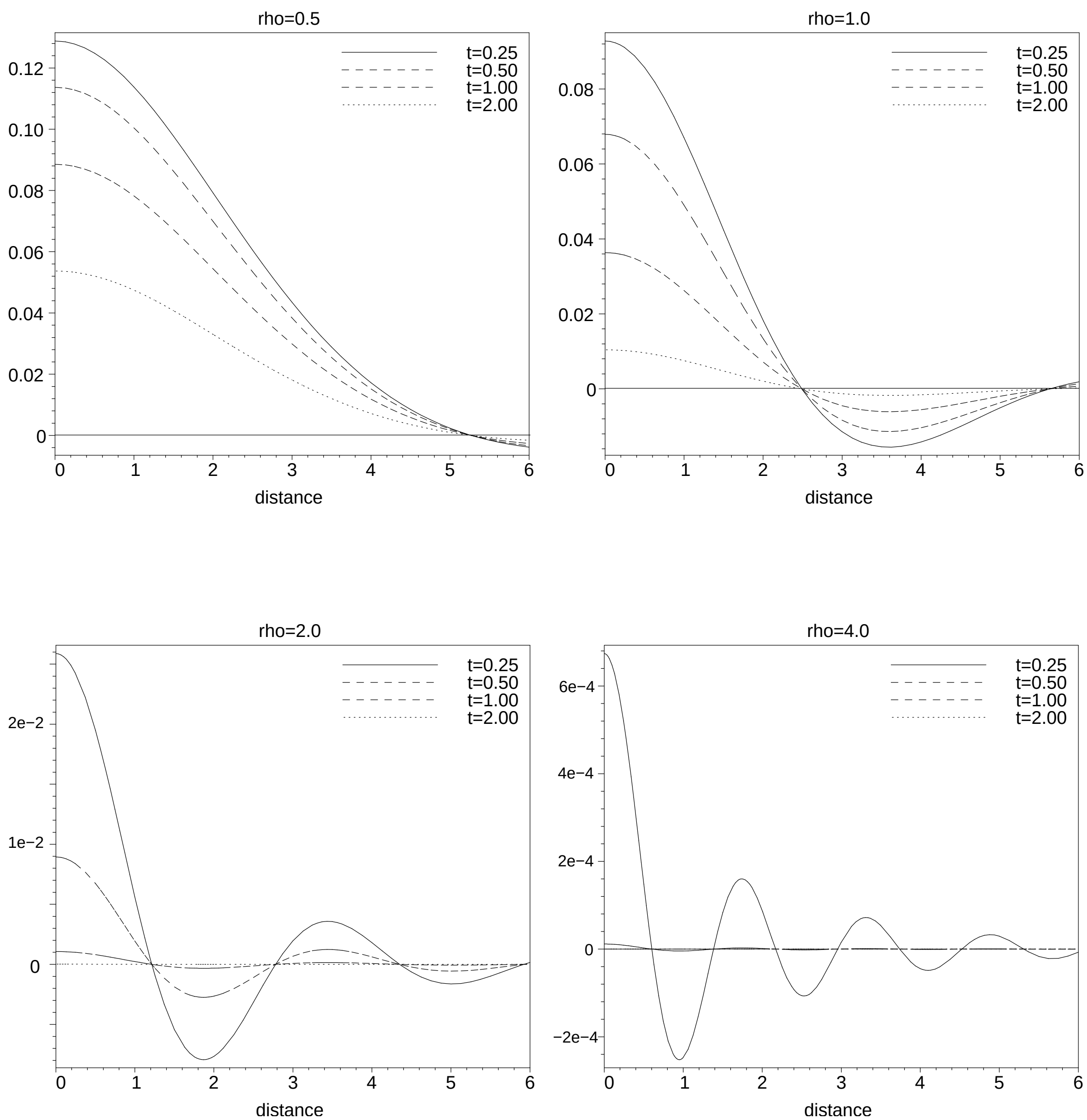

Figure 3.3.1: Graphs of the integrand of $K_0^{(H^2)}$ for various values of ρ.

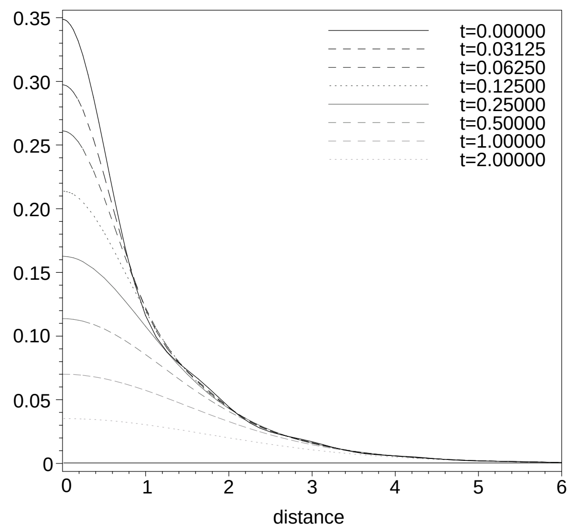

Figure 3.3.2: Graphs of $K_0^{(H^2)}$.

The differential equation, when expanded, becomes

$$\left(-\omega_{\eta\eta} - \frac{\cosh\eta}{\sinh\eta}\omega_\eta + \frac{1}{\sinh^2\eta}\omega + \omega_t\right)\overline{d\eta} = 0.$$

As before, we consider the eigenfunction problem

$$\Delta E = \left(-E_{\eta\eta} - \frac{\cosh\eta}{\sinh\eta}E_\eta + \frac{1}{\sinh^2\eta}E\right) = \lambda E,$$

which, using the transformation on page 46, gives

$$(1-x^2)E_{xx} - 2xE_x + \left(v(v+1) - \frac{1}{1-x^2}\right)E = 0.$$

This is Legendre's equation of order v and degree 1. Thus $E = P^1_{-\frac{1}{2}+i\rho}(\cosh\eta)$, when we go back to the original variables. This function has an eigenvalue of $(\frac{1}{4}+\rho^2)$, and we allow $\rho \in \mathbb{R}^+ \cup \{0\}$. From this it is clear the continuous spectrum of the Laplacian is $[\frac{1}{4}, \infty)$, as stated in [13, 14]. However, there are also harmonic 1-forms on the hyperbolic plane. Thus the full L^2 spectrum of the Laplacian [14] is $\{0\} \cup [\frac{1}{4}, \infty)$. Since the Laplacian is positive as an operator[2], there are no negative

[2] $\langle \Delta f, f\rangle_g = \langle df, df\rangle_g + \langle d^*f, d^*f\rangle_g$.

or complex eigenvalues. The harmonic solutions will not play a role in the solution of the heat equation.

We will make the above eigenfunction into a differential form by writing it as $P^1_{-\frac{1}{2}+i\rho}(\cosh\eta)\overline{d\eta}$. We will use this to create an integral transform.

Let us define

$$\widehat{h}(\rho) := \left\langle h(\eta)\overline{d\eta}, P^1_{-\frac{1}{2}+i\rho}(\cosh\eta)\overline{d\eta}\right\rangle_g$$

which when expanded is

$$\widehat{h}(\rho) = \int_{H^2} h(\eta)P^1_{-\frac{1}{2}+i\rho}(\cosh\eta)\sinh\eta d\eta \wedge d\phi$$

or

$$\widehat{h}(\rho) = 2\pi\int_0^\infty h(\eta)P^1_{-\frac{1}{2}+i\rho}(\cosh\eta)\sinh\eta d\eta.$$

This is proportional [41] to the Mehler-Fock transform of order -1. The inverse of this transform would be

$$h(x)\overline{dx} = -\frac{1}{2\pi}\int_0^\infty \widehat{h}(\rho)P^{-1}_{-\frac{1}{2}+i\rho}(\cosh x)\rho\tanh\pi\rho d\rho \otimes \overline{dx},$$

with the differential forms portion added to maintain consistency with the original problem.

If we apply this transform to the differential equation (3.4.1) we get

$$\left(\frac{1}{4}+\rho^2\right)\widehat{\omega}(\rho,t) + \widehat{\omega}_t(\rho,t) = 0$$

from which we know that

$$\widehat{\omega}(\rho,t) = \widehat{f}(\rho)e^{-\left(\frac{1}{4}+\rho^2\right)t},$$

or

$$\widehat{\omega}(\rho,t) = 2\pi e^{-\left(\frac{1}{4}+\rho^2\right)t}\int_0^\infty f(\eta)P^1_{-\frac{1}{2}+i\rho}(\cosh\eta)\sinh\eta d\eta, \tag{3.4.2}$$

where $f(\eta)\overline{d\eta}$ is the initial condition.

Before we continue, we will state some useful identities (3.4.3 - 3.4.5) from Abramowitz & Stegen [1], and Oberhettinger & Higgins [37]:

$$P^{-1}_{-\frac{1}{2}+i\rho}(\cosh\eta) = -\frac{1}{\frac{1}{4}+\rho^2}\frac{d}{d\eta}P_{-\frac{1}{2}+i\rho}(\cosh\eta) \tag{3.4.3}$$

$$P^{1}_{-\frac{1}{2}+i\rho}(\cosh\eta) = -\left(\frac{1}{4}+\rho^2\right)P^{-1}_{-\frac{1}{2}+i\rho}(\cosh\eta) \tag{3.4.4}$$

$$P_{-\frac{1}{2}+i\rho}(\cosh\eta) = \frac{\cosh\pi\rho}{\pi}\int_1^\infty \frac{\Gamma(1-k)}{(y^2-1)^{\frac{1}{2}k}}\left(\cosh\eta+y\right)^{k-1}P^{k}_{-\frac{1}{2}+i\rho}(y)dy$$
$$(k<\frac{1}{2}) \tag{3.4.5}$$

If we now apply the inverse transform to (3.4.2), we have

$$\begin{aligned}\omega(x,t)\overline{dx} &= -\int_0^\infty e^{-(\frac{1}{4}+\rho^2)t}\left[\int_0^\infty f(\eta)P^{1}_{-\frac{1}{2}+i\rho}(\cosh\eta)\sinh\eta d\eta\right] \\ &\quad\times P^{-1}_{-\frac{1}{2}+i\rho}(\cosh x)\rho\tanh\pi\rho d\rho\otimes\overline{dx}.\end{aligned} \tag{3.4.6}$$

If we convert P^1_ν to P^{-1}_ν as given by (3.4.4), and then use the order-lowering identity (3.4.3) to write P^{-1}_ν as P_ν, we have

$$\begin{aligned}\omega(x,t)\overline{dx} &= \int_0^\infty \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2}\rho\tanh\pi\rho\int_0^\infty f(\eta)\sinh\eta \\ &\quad\times\partial_{\eta x}\left(P_{-\frac{1}{2}+i\rho}(\cosh\eta)P_{-\frac{1}{2}+i\rho}(\cosh x)\right)d\eta d\rho\otimes\overline{dx}.\end{aligned} \tag{3.4.7}$$

By using [26, eq. 8.795.1] with $z_1=\cosh\eta$ and $z_2=\cosh x$, and integrating with respect to ϕ over the interval $[0,2\pi]$, we may write the product of the two Legendre functions as a single Legendre function to give

$$\begin{aligned}\omega(x,t)\overline{dx} &= \frac{1}{2\pi}\int_0^\infty \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2}\rho\tanh\pi\rho\int_0^\infty f(\eta)\sinh\eta \\ &\quad\times\partial_{\eta x}\int_0^{2\pi}P_{-\frac{1}{2}+i\rho}(\cosh\eta\cosh x-\sinh\eta\sinh x\cos\phi)d\phi d\eta d\rho\otimes\overline{dx}.\end{aligned} \tag{3.4.8}$$

By a change of variables, we can introduce θ into the above expression, so that $\phi \mapsto \phi - \theta$. Since the integral with respect to ϕ remove dependence on that variable, and also to θ, the partial derivative, ∂_x and ∂_η, can be replaced with the exterior derivatives $d_{\mathbf{x}}$ and $d_{\mathbf{y}}$, where $\mathbf{x} = (x, \theta)$ and $\mathbf{y} = (\eta, \phi)$. With these two changes, (3.4.8) becomes

$$\begin{aligned}\omega(x,t)\overline{dx} &= \int_0^{2\pi} \left[\int_0^\infty f(\eta) \sinh\eta \left[\frac{1}{2\pi} d_{\mathbf{x}} d_{\mathbf{y}} \int_0^\infty \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2} \rho \tanh \pi\rho \right.\right. \\ &\quad \left.\left. \times\ P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{x},\mathbf{y})) d\rho \right] \right] d\phi. \qquad (3.4.9)\end{aligned}$$

We are able to change the order of the integration and the exterior derivatives because, even though there would now be a $d\theta$ term, the integration with respect to ϕ causes it to vanish[3]. Also note, the second integral is with respect to η, with the $d\eta$ term coming from the $d_{\mathbf{y}}$ exterior derivative. For future work, we will want to rearrange this as

$$\begin{aligned}\omega(x,t)\overline{dx} &= \int_0^{2\pi} \int_0^\infty \left[\frac{1}{2\pi} d_{\mathbf{x}} d_{\mathbf{y}} \int_0^\infty \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2} \rho \tanh \pi\rho P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{x},\mathbf{y})) d\rho \right] \\ &\quad \times f(\eta) \sinh \eta d\phi. \qquad (3.4.10)\end{aligned}$$

Referring back to (3.2.5) we see that if we take as initial conditions $g(\eta)\overline{d\phi}$, the differential equation is the same as the equation for the $\overline{d\eta}$ term. Thus the solution will procede in a similar manner, making slight changes to account for $\overline{d\phi}$ instead of $\overline{d\eta}$. With these modifications, we can write the analogue of (3.4.8) as

$$\begin{aligned}\nu(x,t)\overline{d\theta} &= \frac{1}{2\pi} \int_0^\infty \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2} \rho \tanh \pi\rho \int_0^\infty f(\eta) \sinh \eta \\ &\quad \times \partial_{\eta x} \int_0^{2\pi} P_{-\frac{1}{2}+i\rho}(\cosh\eta\cosh x - \sinh\eta\sinh x\cos\phi) d\phi d\eta d\rho \otimes \overline{d\theta}. \\ &\qquad (3.4.11)\end{aligned}$$

[3]Relevent equations can be found in [26].

As with (3.4.8), we can replace the partial derivatives with exterior derivatives yielding

$$\begin{aligned}\nu(x,t)\overline{d\theta} &= \int_0^{2\pi}\left[\int_0^{\infty} f(\eta)\sinh\eta\left[\frac{1}{2\pi} *_{\mathbf{x}} d_{\mathbf{x}} d_{\mathbf{y}} \int_0^{\infty} \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2}\rho\tanh\pi\rho\right.\right.\\ &\qquad \left.\left.\times\, P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{x},\mathbf{y}))d\rho\right]\right] d\phi. \qquad (3.4.12)\end{aligned}$$

We need the Hodge star operator in terms of $\mathbf{x}$ in order to account for the $\overline{d\theta}$ term. To make things symmetric, we can also apply $*_{\mathbf{y}}$, and write (3.4.12) as

$$\begin{aligned}\nu(x,t)\overline{d\theta} &= \int_0^{\infty}\int_0^{2\pi}\left[\frac{1}{2\pi} *_{\mathbf{x}} *_{\mathbf{y}} d_{\mathbf{x}} d_{\mathbf{y}} \int_0^{\infty} \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2}\rho\tanh\pi\rho\right.\\ &\qquad \left.\times\, P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{x},\mathbf{y}))d\rho\right] f(\eta)d\eta. \qquad (3.4.13)\end{aligned}$$

Now we need consider how to extract the heat kernel from this information. Recall, from the discussion in Remark 2.1.1, the heat kernel gives the solution to the heat equation in the following manner:

$$\omega(\mathbf{x},t) = \langle K(\mathbf{x},\mathbf{y},t), f(\mathbf{y})\rangle_g = \int_M K(\mathbf{x},\mathbf{y},t)\wedge_{\mathbf{y}} *_{\mathbf{y}} f(\mathbf{y}),$$

where the initial condition $f(\mathbf{y})$ is a differential form. With this knowledge, we can rewrite (3.4.10) and (3.4.13) as

$$\begin{aligned}\omega(x,t)\overline{dx} &= \int_0^{2\pi}\int_0^{\infty} K(\mathbf{x},\mathbf{y},t)\wedge_{\mathbf{y}} *_{\mathbf{y}} f(\eta)\overline{d\eta}\\ &= \int_0^{2\pi}\int_0^{\infty} K(\mathbf{x},\mathbf{y},t)\wedge_{\mathbf{y}} f(\eta)\overline{d\phi} \qquad (3.4.14)\\ \nu(x,t)\overline{d\theta} &= \int_0^{2\pi}\int_0^{\infty} K(\mathbf{x},\mathbf{y},t)\wedge_{\mathbf{y}} *_{\mathbf{y}} g(\eta)\overline{d\phi}\\ &= -\int_0^{2\pi}\int_0^{\infty} K(\mathbf{x},\mathbf{y},t)\wedge_{\mathbf{y}} g(\eta)\overline{d\eta}. \qquad (3.4.15)\end{aligned}$$

The negative sign disappears when we switch order in the wedge product. To make

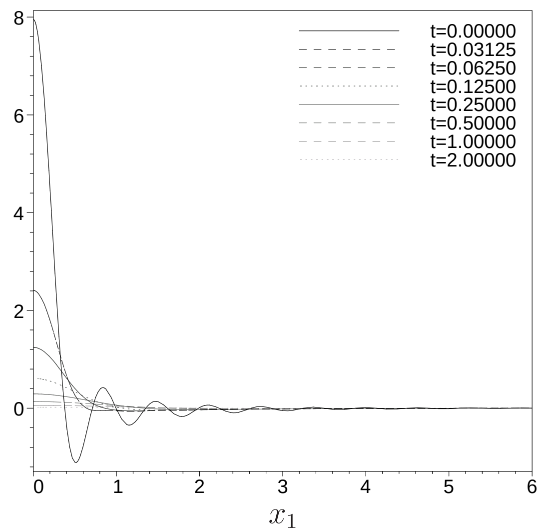

Figure 3.4.1: Graphs of the $dx^1 \otimes dy^1$ portion of $K_1^{(H^2)}$.

$K(\mathbf{x}, \mathbf{y}, t)$ consistent with equations (3.4.10) and (3.4.13), we propose that

$$\begin{aligned} K(\mathbf{x}, \mathbf{y}, t) &= (I + *_{\mathbf{x}} *_{\mathbf{y}})\, d_{\mathbf{x}} d_{\mathbf{y}} \left[\frac{1}{2\pi} \int_0^\infty \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2} \rho \tanh \pi\rho \right. \\ &\qquad \left. \times P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{x}, \mathbf{y})) d\rho \right], \end{aligned} \tag{3.4.16}$$

and we claim that this is the heat kernel on 1-forms for the hyperboloid model of the hyperbolic plane. This claim will be justified in Section 3.5.

In Figure 3.4.1 we show a portion of the 1-form heat kernel, looking at the $dx^1 \otimes dy^1$ component. As with Figure 3.3.2, we have restricted ρ, this time to the interval $[0, 10]$. Since the 1-form heat kernel depends not only on distance, but also its derivatives, we have fixed the $\mathbf{y}$ term to the origin, and have let $\mathbf{x}$ vary.

3.5 Buttig and Eichhorn's Condition

To find the 1-form heat kernel on the hyperbolic plane, we will recall Buttig's definition of a good global heat kernel first conjectured in [6], and its existence and uniqueness was shown in [7].

Definition 3.5.1 *Suppose M is an open and complete N-dimensional manifold of bounded geometry of order up to $k > N/2 + 1$. A double form E^p with values $E^p(\mathbf{x}, \mathbf{y}, t) \in \bigwedge^p T^*_{\mathbf{x}}(M) \otimes \bigwedge^p T^*_{\mathbf{y}}(M)$ is said to be a good global heat kernel, if*

1. *$E^p(\mathbf{x}, \mathbf{y}, t)$ is smooth for $t > 0$,*

2. *$(\Delta_{\mathbf{y}} + \partial_t)E^p(\mathbf{x}, \mathbf{y}, t) = 0$,*

3. *for every $\mathbf{x} \in M$ and ω is a smooth p-form with compact support on M there hold*
$$\int_M E^p(\mathbf{x}, \mathbf{y}, t) \wedge *_{\mathbf{y}}\omega(\mathbf{y}) \to \omega(\mathbf{x})$$
for $t \to 0^+$,

4. *there exist constants $C_1, C_2 > 0$, depending on l, m, n such that for every $0 < t < \infty$, and $\mathbf{x}, \mathbf{y} \in M$*
$$|(\partial_t^l)\nabla^m_{\mathbf{x}}\nabla^n_{\mathbf{y}}E^p(\mathbf{x}, \mathbf{y}, t)| \le c_1 t^{-N/2-(m+n)/2-l} e^{-C_2(d_M(\mathbf{x},\mathbf{y}))^2/t},$$

5. *the heat kernels $E^p(\mathbf{x}, \mathbf{y}, t)$ and $E^{p+1}(\mathbf{x}, \mathbf{y}, t)$ are related by*
$$d_{\mathbf{x}}E^p(\mathbf{x}, \mathbf{y}, t) = d^*_{\mathbf{y}}E^{p+1}(\mathbf{x}, \mathbf{y}, t).$$

Example 3.5.2 We will use property 3.5.1(5) to help us solve for the heat kernel. To demonstrate this property, we will refer to the example of the heat kernel for the polar Euclidean plane, as given in Section 2.3.

From (2.3.1) we recall that
$$K^0(r, \theta; \rho, \phi; t) = \frac{1}{4\pi t} e^{-\dfrac{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}{4t}}$$

which means that

$$K^2(r,\theta;\rho,\phi;t) = \frac{1}{4\pi t} e^{-\dfrac{r^2+\rho^2-2r\rho\cos(\theta-\phi)}{4t}} r\rho dr \wedge d\theta \otimes d\rho \wedge d\phi.$$

From (2.3.6, 2.3.7) we know that

$$\begin{aligned} K^1(r,\theta;\rho,\phi;t) = \frac{1}{4\pi t} e^{-\dfrac{r^2+\rho^2-2r\rho\cos(\theta-\phi)}{4t}} & \\ \times \big(\cos(\theta-\phi)\overline{dr} \otimes \overline{d\rho} + \sin(\theta-\phi)\overline{dr} \otimes \overline{d\phi} & \\ - \sin(\theta-\phi)\overline{d\theta} \otimes \overline{d\rho} + \cos(\theta-\phi)\overline{d\theta} \otimes \overline{d\phi}\big) & \end{aligned}$$

We will show that $d_{\mathbf{x}} K^1 = d^*_{\mathbf{y}} K^2$.

Note that we can write $K^2 = K^0 r\rho dr d\theta \otimes d\rho \wedge d\phi$, and $K^1 = K^0\Omega$, where Ω is given in (2.3.7). Because of the exponential form of K^0, we can factor K^0 out of the exterior derivative and coderivative to make the equations simpler.

First,

$$\begin{aligned} d^*_{\mathbf{y}} K^2 &= - * d_{\mathbf{y}} * K^2 \\ &= - * d_{\mathbf{y}} K^0 \\ &= - * K^0 \left(-\frac{1}{2t} \left((2\rho - 2r\cos(\theta-\phi))\, d\rho - 2r\rho\sin(\theta-\phi) d\phi \right) \right) \\ &= \frac{1}{2t} K^0 \left(r\sin(\theta-\phi) r dr \wedge d\theta \otimes d\rho + (\rho - r\cos(\theta-\phi))\rho r dr \wedge d\theta \otimes d\phi \right). \end{aligned}$$

Then,

$$\begin{aligned}
d_{\mathbf{x}}K1 &= K^0\left(d_{\mathbf{x}}\left(-\frac{r^2+\rho^2-2r\rho\cos(\theta-\phi)}{4t}\right)\Omega - \frac{r^2+\rho^2-2r\rho\cos(\theta-\phi)}{4t}d_{\mathbf{x}}\Omega\right)\\
&= -\frac{1}{4t}K^0\left(\Omega\left((2r-2\rho\cos(\theta-\phi))dr + 2r\rho\sin(\theta-\phi)d\theta\right)\right)\\
&= \frac{1}{2t}K^0\left[(r\sin(\theta-\phi)(r-\rho\cos(\theta-\phi))\right.\\
&\qquad + r\rho\sin(\theta-\phi)\cos(\theta-\phi))dr\wedge d\theta\otimes d\rho\\
&\qquad - \left.(r\rho\cos(\theta-\phi)(r-\rho\cos(\theta-\phi)) - r\rho^2\sin^2(\theta-\phi))dr\wedge d\theta\otimes d\phi\right]\\
&= \frac{1}{2t}K^0\left(r^2\sin(\theta-\phi)dr\wedge d\theta\otimes d\rho - r\rho(r\cos(\theta-\phi)-\rho)dr\wedge d\theta\otimes d\phi\right)
\end{aligned}$$

Clearly, we have equality. †

Because of the symmetry of the spatial variables in the heat kernel, arising from properties of Green's functions, we can also write property 5 above as $d_{\mathbf{y}}E^p = d^*_{\mathbf{x}}E^{p+1}$.

The 1-form heat kernel on a Riemann surface will have the form

$$\begin{aligned}
K_1(\mathbf{x},\mathbf{y},t) &= A_{11}(\mathbf{x},\mathbf{y},t)dx^1\otimes dy^1 + A_{12}(\mathbf{x},\mathbf{y},t)dx^1\otimes dy^2\\
&\quad + A_{21}(\mathbf{x},\mathbf{y},t)dx^2\otimes dy^1 + A_{22}(\mathbf{x},\mathbf{y},t)dx^2\otimes dy^2 \qquad (3.5.1)
\end{aligned}$$

which we will rewrite as

$$K_1 = \left(A_{11}dx^1 + A_{21}dx^2\right)\otimes dy^1 + \left(A_{12}dx^1 + A_{22}dx^2\right)\otimes dy^2.$$

We can use noncompact Hodge decomposition, [8] to write

$$\begin{aligned}
A_{11}dx^1 + A_{21}dx^2 &= d_{\mathbf{x}}\omega + d^*_{\mathbf{x}}\nu dx^1\wedge dx^2 + h_1\\
A_{12}dx^1 + A_{22}dx^2 &= d_{\mathbf{x}}\alpha + d^*_{\mathbf{x}}\beta dx^1\wedge dx^2 + h_2 \qquad (3.5.2)
\end{aligned}$$

where $\alpha,\omega\in\bigwedge^0 T^*M$, $\nu dx^1\wedge dx^2, \beta dx^1\wedge dx^2\in\bigwedge^2 T^*M$ and h_1, h_2 are harmonic 1-forms, meaning $\Delta_{\mathbf{x}}h_i = 0$, or equivalently, $d_{\mathbf{x}}h_i = d^*_{\mathbf{x}}h_i = 0$. Since we are planning to use Buttig and Eichhorn's condition, the harmonic portions vanish for this part of the calculation.

We will use the following two forms of the Buttig and Eichhorn condition:

$$d_{\mathbf{x}} K_1 = d^*_{\mathbf{y}} K_2 \tag{3.5.3}$$

$$d^*_{\mathbf{x}} K_1 = d_{\mathbf{y}} K_0 \tag{3.5.4}$$

which we can write, using the Hodge decomposition above, as

$$d_{\mathbf{x}} d^*_{\mathbf{x}} \nu dx^1 \wedge dx^2 \otimes dy^1 + d_{\mathbf{x}} d^*_{\mathbf{x}} \beta dx^1 \wedge dx^2 \otimes dy^2 = - *_{\mathbf{y}} d_{\mathbf{y}} *_{\mathbf{y}} K_2 \tag{3.5.5}$$

$$d^*_{\mathbf{x}} d_{\mathbf{x}} \omega \otimes dy^1 + d^*_{\mathbf{x}} d_{\mathbf{x}} \alpha \otimes dy^2 = d_{\mathbf{y}} K_0 \tag{3.5.6}$$

Since $K_2 = *_{\mathbf{x}} *_{\mathbf{y}} K_0$ (§2.2), we can write (3.5.5) as

$$d_{\mathbf{x}} d^*_{\mathbf{x}} \nu dx^1 \wedge dx^2 \otimes dy^1 + d_{\mathbf{x}} d^*_{\mathbf{x}} \beta dx^1 \wedge dx^2 \otimes dy^2 = - *_{\mathbf{x}} *_{\mathbf{y}} d_{\mathbf{y}} K_0 \tag{3.5.7}$$

By applying the Hodge star in $\mathbf{x}$ to both sides of (3.5.7), we can combine (3.5.6) and (3.5.7) to give

$$d^*_{\mathbf{x}} d_{\mathbf{x}} *_{\mathbf{x}} \left(\nu dx^1 \wedge dx^2 \otimes dy^1 + \beta dx^1 \wedge dx^2 \otimes dy^2 \right) = - *_{\mathbf{y}} d^*_{\mathbf{x}} d_{\mathbf{x}} \left(\omega \otimes dy^1 + \alpha \otimes dy^2 \right) \tag{3.5.8}$$

Both sides of (3.5.8) have $d^*_{\mathbf{x}} d_{\mathbf{x}}$ acting on forms, in particular, it is acting on 0-forms in $\mathbf{x}$. We recall that $d^*_{\mathbf{x}} d_{\mathbf{x}}$ is the Laplacian on 0-forms. Also, since the Hodge star on the right-hand side of (3.5.8) is with respect to $\mathbf{y}$, it will commute with the Laplacian. Thus if we have a general symmetric Riemannian metric, g_{ij}, we can write

$$\begin{aligned} \Delta^{(0)}_{\mathbf{x}} \Big(g(\mathbf{y})^{-\frac{1}{2}} \left(\omega \otimes \left[g_{12}(\mathbf{y}) dy^1 + g_{22}(\mathbf{y}) dy^2 \right] \right. & - \left. \alpha \otimes \left[g_{11}(\mathbf{y}) dy^1 + g_{21}(\mathbf{y}) dy^2 \right] \right) \\ + g(\mathbf{x})^{-\frac{1}{2}} \left(\nu \otimes dy^1 + \beta \otimes dy^2 \right) \Big) & = 0 \end{aligned} \tag{3.5.9}$$

Since there are no non-zero L^2 bounded harmonic functions on H^2, which is the case we are considering, (3.5.9) tell us that the argument of the Laplacian must be zero. This means

$$g(\mathbf{x})^{-\frac{1}{2}}\nu + g(\mathbf{y})^{-\frac{1}{2}}\left(\omega g_{12}(\mathbf{y}) - \alpha g_{11}(\mathbf{y})\right) = 0 \tag{3.5.10}$$

$$g(\mathbf{x})^{-\frac{1}{2}}\beta + g(\mathbf{y})^{-\frac{1}{2}}\left(\omega g_{22}(\mathbf{y}) - \alpha g_{21}(\mathbf{y})\right) = 0 \tag{3.5.11}$$

This means we can solve for ν and β in terms of α and ω.

From (3.5.6) we know that

$$\Delta^{(0)}_{\mathbf{x}}\omega = \partial_{y^1}K_0 \tag{3.5.12}$$

$$\Delta^{(0)}_{\mathbf{x}}\alpha = \partial_{y^2}K_0 \tag{3.5.13}$$

Thus, if we can solve a Dirac equation, as given above, we will be able to write the 1-form heat kernel in terms of the 0-form heat kernel. For this, we will use the upper half-plane model of the hyperbolic plane. But first, we will show some simplifications of the presentation of the heat kernel.

By solving (3.5.10) and (3.5.11) for ν and β and substituting into (3.5.1) we find

$$\begin{aligned} K_1 \;=\; & \left(d_{\mathbf{x}}\omega + g(\mathbf{y})^{-\frac{1}{2}} *_{\mathbf{x}} d_{\mathbf{x}}\left(\omega g_{12}(\mathbf{y}) - \alpha g_{11}(\mathbf{y})\right)\right) \otimes dy^1 \\ & \left(d_{\mathbf{x}}\alpha + g(\mathbf{y})^{-\frac{1}{2}} *_{\mathbf{x}} d_{\mathbf{x}}\left(\omega g_{22}(\mathbf{y}) - \alpha g_{21}(\mathbf{y})\right)\right) \otimes dy^2 \end{aligned} \tag{3.5.14}$$

If the metric is diagonal, as is the case for the hyperbolic and euclidean planes, this expression simplifies, using some properties of the Hodge star, to

$$K_1 = [I + *_{\mathbf{x}} *_{\mathbf{y}}]\left(\left[d_{\mathbf{x}}\omega - \left(\frac{g_{11}(\mathbf{y})}{g_{22}(\mathbf{y})}\right)^{\frac{1}{2}} *_{\mathbf{x}} d_{\mathbf{x}}\alpha\right] \otimes dy^1\right). \tag{3.5.15}$$

Since we can write $\omega = \left[\Delta^{(0)}_{\mathbf{x}}\right]^{-1}\partial_{y^1}K_0$, and $\alpha = \left[\Delta^{(0)}_{\mathbf{x}}\right]^{-1}\partial_{y^2}K_0$, where $\left[\Delta^{(0)}_{\mathbf{x}}\right]^{-1}$ is the solution operator for Poisson's equation, most of the work needs to be concentrated on finding $Q = \left[\Delta_{\mathbf{x}}^{(0)}\right]^{-1} K_0$. With this, we write,

$$K_1 = \left([I + *_{\mathbf{x}} *_{\mathbf{y}}] \left[d_{\mathbf{x}} \partial_{y^1} - \left(\frac{g_{11}(\mathbf{y})}{g_{22}(\mathbf{y})} \right)^{\frac{1}{2}} *_{\mathbf{x}} \, d_{\mathbf{x}} \partial_{y^2} \right] \right) \left(Q \otimes dy^1 \right). \qquad (3.5.16)$$

If we expand this expression, and rearrange, we can write the 1-form heat kernel for a surface with diagonal metric, as

$$K_1(\mathbf{x}, \mathbf{y}, t) = (I + *_{\mathbf{x}} *_{\mathbf{y}}) \, d_{\mathbf{x}} d_{\mathbf{y}} \left[\Delta_{\mathbf{x}}^{(0)}\right]^{-1} K_0(\mathbf{x}, \mathbf{y}, t). \qquad (3.5.17)$$

It is straightforward to verify that (3.5.17) satifies the heat equation.

$$\begin{aligned} \Delta_{\mathbf{x}}^{(1)} K_1 &= (I + *_{\mathbf{x}} *_{\mathbf{y}}) \, d_{\mathbf{y}} \Delta_{\mathbf{x}}^{(1)} d_{\mathbf{x}} \left[\Delta_{\mathbf{x}}^{(0)}\right]^{-1} K_0 \\ &= (I + *_{\mathbf{x}} *_{\mathbf{y}}) \, d_{\mathbf{y}} d_{\mathbf{x}} \Delta_{\mathbf{x}}^{(0)} \left[\Delta_{\mathbf{x}}^{(0)}\right]^{-1} K_0 \\ &= (I + *_{\mathbf{x}} *_{\mathbf{y}}) \, d_{\mathbf{y}} d_{\mathbf{x}} K_0 \end{aligned}$$

and

$$\begin{aligned} \partial_t K_1 &= (I + *_{\mathbf{x}} *_{\mathbf{y}}) \, d_{\mathbf{y}} d_{\mathbf{x}} \left[\Delta_{\mathbf{x}}^{(0)}\right]^{-1} \partial_t K_0 \\ &= -(I + *_{\mathbf{x}} *_{\mathbf{y}}) \, d_{\mathbf{y}} d_{\mathbf{x}} \left[\Delta_{\mathbf{x}}^{(0)}\right]^{-1} \Delta_{\mathbf{x}}^{(0)} K_0 \\ &= -(I + *_{\mathbf{x}} *_{\mathbf{y}}) \, d_{\mathbf{y}} d_{\mathbf{x}} K_0. \end{aligned}$$

Thus we have $\Delta_{\mathbf{x}}^{(1)} K_1 + \partial_t K_1 = 0$. Now referring back to the harmonic portions of (3.5.2), we can now argue that the harmonic 1-forms are zero. The reason is as follows: since the expression above satifies the heat equation, and the 1-forms h_1, h_2 are harmonic, we must have that h_1 and h_2 are independent of t. However, since the heat kernel must go to zero pointwise for large time, we have that h_1 and h_2 are zero.

In (3.5.17), we would like to rewrite the operator $(\Delta_{\mathbf{x}}^{(0)})^{-1}$ to make the expression more tractable. To this end, we will show some properties of this operator, which we will write as $G_0(\mathbf{x})$. First, we have $G_0(\mathbf{x})e^{-t\Delta_{\mathbf{x}}^{(0)}} = e^{-t\Delta_{\mathbf{x}}^{(0)}}G_0(\mathbf{x})$, which can be shown through symbolic power series expansion of the heat equation solution operator or through the functional calculus. Next, we have

$$\begin{aligned} G_0(\mathbf{x})\int_M K_0(\mathbf{x},\mathbf{y},t)\wedge_{\mathbf{y}} *_{\mathbf{y}} f(\mathbf{y}) &= G_0(\mathbf{x})e^{-t\Delta_{\mathbf{x}}^{(0)}} f(\mathbf{x}) \\ &= e^{-t\Delta_{\mathbf{x}}^{(0)}} G_0(\mathbf{x}) f(\mathbf{x}) \\ &= \int_M K_0(\mathbf{x},\mathbf{y},t)\wedge_{\mathbf{y}} *_{\mathbf{y}} G_0(\mathbf{y}) f(\mathbf{y}) \quad (3.5.18) \end{aligned}$$

From this, we find

$$\begin{aligned} \partial_t G_0(\mathbf{x})K_0(\mathbf{x},\mathbf{y},t) &= \partial_t G_0(\mathbf{x})\int_M K_0(\mathbf{x},\mathbf{z},0)\wedge_{\mathbf{z}} *_{\mathbf{z}} K_0(\mathbf{z},\mathbf{y},t) \\ &= G_0(\mathbf{x})\int_M K_0(\mathbf{x},\mathbf{z},0)\wedge_{\mathbf{z}} *_{\mathbf{z}} \partial_t K_0(\mathbf{z},\mathbf{y},t) \\ &= -G_0(\mathbf{x})\int_M K_0(\mathbf{x},\mathbf{z},0)\wedge_{\mathbf{z}} *_{\mathbf{z}} \Delta_{\mathbf{x}}^{(0)} K_0(\mathbf{z},\mathbf{y},t) \\ &= -\int_M K_0(\mathbf{x},\mathbf{z},0)\wedge_{\mathbf{z}} *_{\mathbf{z}} G_0(\mathbf{z})\Delta_{\mathbf{z}}^{(0)} K_0(\mathbf{z},\mathbf{y},t) \\ &= -\int_M K_0(\mathbf{x},\mathbf{z},0)\wedge_{\mathbf{z}} *_{\mathbf{z}} K_0(\mathbf{z},\mathbf{y},t) \\ &= -K_0(\mathbf{x},\mathbf{y},t). \end{aligned} \quad (3.5.19)$$

This means we can write $G_0(\mathbf{x})K_0(\mathbf{x},\mathbf{y},t) = -\int_\infty^t K_0(\mathbf{x},\mathbf{y},\tau)d\tau = \int_t^\infty K_0(\mathbf{x},\mathbf{y},\tau)d\tau$. Therefore, (3.5.17) can be written as

$$K_1(\mathbf{x},\mathbf{y},t) = (I + *_{\mathbf{x}} *_{\mathbf{y}})\, d_{\mathbf{x}} d_{\mathbf{y}} \int_t^\infty K_0(\mathbf{x},\mathbf{y},\tau)d\tau. \quad (3.5.20)$$

For the case of the hyperbolic plane, how does this compare with (3.4.16)? From (3.3.3) we have the 0-form heat kernel

$$K_0^{(H^2)}(\mathbf{z},\mathbf{w},t) = (2\pi)^{-1}\int_0^\infty P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{z},\mathbf{w}))\rho e^{-(\frac{1}{4}+\rho^2)t}\tanh \pi\rho d\rho,$$

which, if we integrate with respect to the time variable, we get

$$\int_t^\infty K_0^{(H^2)}(\mathbf{z},\mathbf{w},\tau)d\tau = (2\pi)^{-1}\int_0^\infty P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{z},\mathbf{w}))\rho \frac{e^{-(\frac{1}{4}+\rho^2)t}}{\frac{1}{4}+\rho^2}\tanh \pi\rho d\rho.$$

This is exactly the result we need for the proposed solution at the end of Section 3.4.

Example 3.5.3 Let us apply (3.5.20) to the heat kernel for the Cartesian plane, as given in (2.1.6). To do this, we interchange the exterior derivatives and the integration and then evaluate.

$$
\begin{aligned}
K_1(\mathbf{x},\mathbf{y},t) &= \int_t^\infty (I + *_{\mathbf{x}}*_{\mathbf{y}})\, d_{\mathbf{x}} d_{\mathbf{y}} \frac{1}{4\pi\tau} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\tau}} d\tau \\
&= \int_t^\infty (I + *_{\mathbf{x}}*_{\mathbf{y}}) \\
&\qquad \times \frac{1}{8\pi\tau^2} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\tau}} \left[-\frac{1}{2\tau} \left((x_1-y_1)^2 dx^1 \otimes dy^1 \right.\right. \\
&\qquad + (x_1-y_1)(x_2-y_2) \left(dx^1 \otimes dy^2 + dx^2 \otimes dy^1 \right) \\
&\qquad \left.\left. + (x_2-y_2)^2 dx^2 \otimes dy^2 \right) + dx^1 \otimes dy^1 + dx^2 \otimes dy^2 \right] d\tau \\
&= \frac{1}{8\pi} \int_t^\infty e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\tau}} \left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{2\tau^3} + 2 \right] \left(dx^1 \otimes dy^1 + dx^2 \otimes dy^2 \right) d\tau \\
&= \left[-\frac{1}{\pi|\mathbf{x}-\mathbf{y}|^2} + \frac{e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} 4\pi t}{+} \frac{e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}}}{\pi|\mathbf{x}-\mathbf{y}|^2} + \frac{1}{\pi|\mathbf{x}-\mathbf{y}|^2} - \frac{e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}}}{\pi|\mathbf{x}-\mathbf{y}|^2} \right] \\
&\qquad \times \left(dx^1 \otimes dy^1 + dx^2 \otimes dy^2 \right) \\
&= \frac{1}{4\pi t} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} \left(dx^1 \otimes dy^1 + dx^2 \otimes dy^2 \right) \qquad (3.5.21)
\end{aligned}
$$

We can see that this agrees with the expression given in Remark 2.3.1. †

Chapter 4
Tilings

4.1 The Uniformization Theorem

We will use the heat kernel found in the previous chapter to generate the heat kernel for Riemann surfaces in general, using the hyperbolic plane as a covering space for the Riemann surfaces. First, we will define what we mean by a universal cover, and then give some examples.

Definition 4.1.1 *[36] Let $p : M \to N$ be continuous and onto, and suppose that for every point $\mathbf{x} \in N$, there is an open set $U_{\mathbf{x}} \subset N$, whose inverse image $p^{-1}(U_{\mathbf{x}})$ is a union of disjoint open sets, $\{V_i\}$, in M, such that V_i is homeomorphic to $U_{\mathbf{x}}$. Then M is called a covering space of N, and the map p is called a covering map. If the space M is simply connected, it is called a universal cover.*

Example 4.1.2 We will illustrate covering spaces with an example. Consider the unit circle, S^1, and the real line, $\mathbb{R}$. The map $p : \mathbb{R} \to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is continuous and onto. For an illustration, see Figure 4.1.1. Furthermore, any open set in S^1 corresponds to an infinite collection of open intervals in $\mathbb{R}$. †

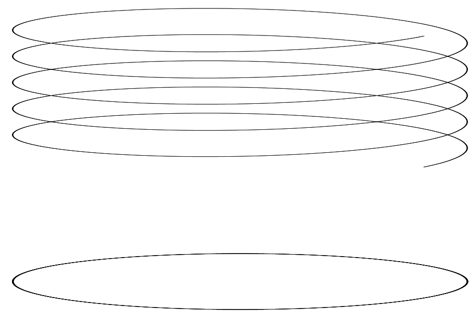

Figure 4.1.1: Covering of S^1

Theorem 4.1.3 *[2, 21]* **[The Uniformization Theorem]** *Let M be a simply connected Riemann surface. Then there is a biholomorphic function $\phi : M \to U$, where U is either S^2, $\mathbb{R}^2$, or H^2, that is the sphere, the plane, or the hyperbolic plane.*

We should note that, with a few exceptions, the hyperbolic plane is the universal cover for any Riemann surface.

Definition 4.1.4 *[2, 38] Let M be a Riemann surface, and let U be its universal covering space. Then there is a map $p : U \to M$ satisfying (4.1.1). Let A be the group of automorphisms of U. We say $g \in A$ is a covering transformations of M if $p(g \cdot \mathbf{x}) = p(\mathbf{x})$. The group of covering transformations is called the covering group of M. Given a covering group, G, we say $R \subseteq U$ is a fundamental region of M if*

1. *R is open in U,*
2. *$gR \cap hR = \emptyset$ if $g \neq h \in G$, and*
3. *$U = \cup_{g \in G}\, g\overline{R}$.*

Theorem 4.1.5 *[2] Let M be a Riemann surface and U be a universal cover of M. Let G be the covering group of M. Then M is conformally equivalent to U/G.*

Example 4.1.6 The following are some simple examples of Riemann surfaces as quotients of the universal cover.

1. The torus, $T_1 = \mathbb{R}^2/\mathbb{Z}^2$. The fundamental region is the unit square.

2. The cylinder, $C = \mathbb{R}^2/\mathbb{Z}$. The fundamental region is an infinitely long strip, one unit in width.

3. The cone, while not a Riemann surface, can be thought of as $\mathbb{R}^2/\mathbb{Z}_n$. The fundamental region is an infinite wedge with central angle of $\theta = \frac{2\pi}{n}$.

4. The 2-holed torus, $T_2 = H^2/G$. The fundamental region is an octagon in H^2, with sides identified as in Figure 4.1.2. The group G is the group on four

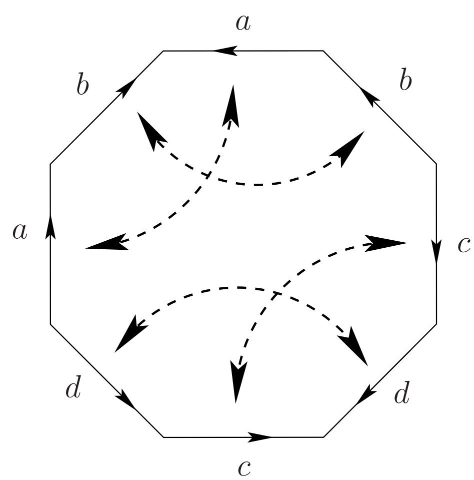

Figure 4.1.2: Fundamental domain for the double torus.

generators a, b, c, d subject to the relation $aba^{-1}b^{-1}cdc^{-1}d^{-1} = e$. For further information about the action of this group on H^2, see [22, 31].

5. We should also consider the hyperbolic cylinder and cone: $H^2/\mathbb{Z}$, and $H^2/\mathbb{Z}_n$. To get the hyperbolic cylinder, the group acting on the hyperbolic plane is most easily described using the upper half-plane model, where $n \cdot (x, y) = (x + n, y)$.

In the case of the hyperbolic cone, the group action is best described in the hyperboloid model, where $k \cdot (r, \theta) = (r, \theta + \frac{2\pi k}{n})$.

$\dagger$

We plan to use the heat kernel on H^2 to compute the heat kernels on these quotients by lifting the surface to universal cover, solving the heat equation there, then projecting the solution back down to the surface. This will give us a sum over isometries in G for the heat kernel on the surface. Generally this will not simplify, but may aid in computation.

4.2 Tilings and Heat Kernels

Definition 4.2.1 *Let M be a manifold, and G a group which acts on the manifold. A function, f, on the manifold is said to be G-periodic if $f(g \cdot \mathbf{x}) = f(\mathbf{x})$ for each $g \in G$ and $\mathbf{x} \in M$.*

Example 4.2.2 The function sin : $\mathbb{R} \to \mathbb{R}$ is $\mathbb{Z}$-periodic, where $\mathbb{Z}$ acts on $\mathbb{R}$ by $n \cdot x = x + 2\pi n$. $\dagger$

Proposition 4.2.3 *Let M be a manifold, U be the universal cover of M with covering group G, and let V be a vector space. Suppose we have a G-periodic function $f : U \to V$. Then there is a unique function $\widehat{f} : M \to V$ such that $f = \widehat{f} \circ p$, where p is the covering map from U to M.*

Proof: From 4.1.4 we know that if $\mathbf{y} = p(\mathbf{x})$, then $p^{-1}(\mathbf{y}) = \{g \cdot \mathbf{x} | g \in G\}$. Let f be G-periodic. We will define $\widehat{f}$ as follows: $\widehat{f}(\mathbf{y}) = f(p^{-1}(\mathbf{y}))$. The inverse image of the point $\mathbf{y}$ is a set of points in U. However, as stated above, each of those points is the image of one point under the action of the group G. Since f is G-periodic, the set $p^{-1}(\mathbf{y})$ is mapped to a single point, making the function $\widehat{f}$ well-defined.

To show that $\widehat{f}$ is unique, assume that $f = \widehat{f} \circ p = \widehat{g} \circ p$. Since p is onto, $p(U) = M$, hence $\widehat{f}(\mathbf{x}) = \widehat{g}(\mathbf{x})$ for all $\mathbf{x} \in M$. ■

Theorem 4.2.4 *Let G be a group of isometries. The solution of the heat equation with G-periodic initial conditions is G-periodic.*

Proof:

$$\begin{aligned} u(g \cdot \mathbf{x}, t) &= \int_U K(g \cdot \mathbf{x}, \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} \\ &= \int_U K(g \cdot \mathbf{x}, g \cdot \mathbf{y}, t) f(g \cdot \mathbf{y}) d\mathbf{y} \\ &= \int_U K(g \cdot \mathbf{x}, g \cdot \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} \\ &= \int_U K(\mathbf{x}, \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} \\ &= u(\mathbf{x}, t) \end{aligned}$$

■

Theorem 4.2.4 shows that the solution operator for the heat equation preserves the G-periodic property of the initial conditions, where G is a group of isometries. Given a manifold M and its universal cover U, any function on M can be lifted to a G-periodic function on U, where G is the covering group. By lifting the initial conditions of the heat equation to the universal cover, we can solve the heat equation on the universal cover and then project the solution back down to the orginal manifold. With this construction, we can also write an expression for the heat kernel of the manifold M in terms of the heat kernel of the universal cover. We will develop this method in the next two sections, giving examples, first in the case of $\mathbb{R}^2$, and then for H^2.

4.3 Examples: The Torus, Cylinder, and Cone

For some simple examples we will find the heat kernels on the torus, the cylinder, and the cone. Using the tiling method described later in this manuscript, we find the kernels for these surfaces.

Here we will outline how we solved for the torus heat kernel on functions, and then we will state the results for forms and for the cylinder. The heat kernel on the cone will be slightly easier since it is a finite tiling, indicating a finite sum. We choose for our example the simplest cone, with a tiling by a half-plane. This will be an important example since it uses the polar coordinate form of the heat kernel on the plane, which gives mixed terms in the 1-form case.

We are going to consider a torus, T_1, as a quotient of $\mathbb{R}^2$ by the fundamental group of the torus, $\mathbb{Z} \times \mathbb{Z}$. Since we can parameterize the torus with two angle variables, we tile $\mathbb{R}^2$ with squares of sides length 1. The group acts on $\mathbb{R}^2$ by vector addition, that is,

$$(n, m) \cdot (x, y) = (x + n, y + m)$$

with $\cdot$ indicating group action.

Let us consider how the scalar heat equation lifts from T_1 to $\mathbb{R}^2$. If $f(x, y)$ is the initial condition on T_1, we can extend that to a doubly-periodic function, $\hat{f}$ on all $\mathbb{R}^2$. It can be demonstrated that solving the heat equation on the torus is equivalent to solving the lifted heat equation on $\mathbb{R}^2$ with periodic initial conditions.

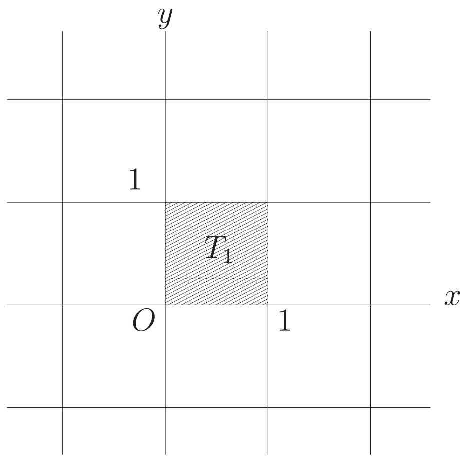

Figure 4.3.1: The fundamental domain of the torus.

As stated previously, the solution to the heat equation on any surface can be written

$$u(\mathbf{x}, t) = \int_M K(\mathbf{x}, \mathbf{y}, t) \wedge *f(\mathbf{y}).$$

The case we need to consider is 0-forms over $\mathbb{R}^2$. We suppose that we know what the heat kernel is, and that we know how to integrate. For ease of reading, we will not substitute the full expression for $K = K_0^{(\mathbb{R}^2)}$.

$$\begin{aligned}
u(\mathbf{x}, t) &= \int_{\mathbb{R}^2} K(\mathbf{x}, \mathbf{y}, t) \wedge *\hat{f}(\mathbf{y}) \\
&= \sum_{n,m \in \mathbb{Z}} \int_{(n,m)\cdot T} K(\mathbf{x}, \mathbf{y}, t) \wedge *\hat{f}(\mathbf{y}) \\
&= \sum_{n,m \in \mathbb{Z}} \int_T K(\mathbf{x}, (n,m) \cdot \mathbf{y}, t) \wedge *\hat{f}((n,m) \circ \mathbf{y}) \\
&= \int_T \sum_{n,m \in \mathbb{Z}} K(\mathbf{x}, (n,m) \cdot \mathbf{y}, t) \wedge *f(\mathbf{y})
\end{aligned}$$

We define the heat kernel for 0-forms on the torus as

$$K_T = \sum_{n,m \in \mathbb{Z}} K(\mathbf{x}, (n,m) \cdot \mathbf{y}, t),$$

recalling from above, that K is the 0-form Euclidean heat kernel.

This process for finding the heat kernel for a surface defined as a quotient works similarly for the hyperbolic plane.

We should say something about the convergence of the heat kernel of the torus. Previously, (equation 2.1.6), we saw that $K_0 \sim e^{\frac{-k^2}{t}}$, and as $(n, m) \cdot \mathbf{y}$ gets further from $\mathbf{x}$, the value of k grows. Thus, the sum converges pointwise for $t > 0$.

As we discussed in section 2.2, the heat kernels on $\mathbb{R}^2$, K_1 and K_2, depend on K_0. Thus we can easily get the heat kernel for forms on the torus.

Define the function

$$K(\mathbf{x}, \mathbf{y}, t) = (4\pi t)^{-1} e^{-|\mathbf{x}-\mathbf{y}|^2/(4t)}.$$

Then the heat kernels on $\mathbb{R}^2$ with the usual coordinates are

$$K_0^{(\mathbb{R}^2)}(\mathbf{x}, \mathbf{y}, t) = K(\mathbf{x}, \mathbf{y}, t) 1_{\mathbf{x}} \otimes 1_{\mathbf{y}}$$

$$K_1^{(\mathbb{R}^2)}(\mathbf{x}, \mathbf{y}, t) = K(\mathbf{x}, \mathbf{y}, t)(dx_1 \otimes dy_1 + dx_2 \otimes dy_2)$$

$$K_2^{(\mathbb{R}^2)}(\mathbf{x}, \mathbf{y}, t) = K(\mathbf{x}, \mathbf{y}, t) dx_1 \wedge dx_2 \otimes dy_1 \wedge dy_2.$$

If we take the fundamental domain of the torus in $\mathbb{R}^2$ to be the unit square, we know that $\mathbb{Z}^2$ acts in the following manner: $(n, m) \cdot \mathbf{x} = (x_1 + n, x_2 + m)$. Following an argument in Section 4.2, we can write

$$K_i^{(T)}(\mathbf{x}, \mathbf{y}, t) = \sum_{n,m \in \mathbb{Z}} K_i^{(\mathbb{R}^2)}(\mathbf{x}, (n, m) \cdot \mathbf{y}, t).$$

Let us consider $i = 0$ in the above, and ignore for the moment the "forms" portion of the statement. That means we would like to calculate

$$T(z, t) = \sum_{n \in \mathbb{Z}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(z-n)^2}{4t}},$$

since then we would have that

$$K_0^{(T)}(\mathbf{x}, \mathbf{y}, t) = T(x_1 - y_1, t)T(x_2 - y_2, t).$$

By applying Poisson's summation formula [17] which says that if we have a smooth, rapidly decreasing function, $f(x)$, on $\mathbb{R}$ then the sum over the integers of $f(n)$ is equal to the sum, also over the integers, $\widehat{f}(w)$ of the Fourier transformed function. Then we have a formula from Abramowitz and Stegun [1],

$$T(z,t) = \sum_{w \in \mathbb{Z}} e^{-wzi} e^{-w^2 t} = 1 + 2\sum_{w=1}^{\infty} e^{-w^2 t} \cos wz = \theta_3(z/2, e^{-t}),$$

where θ_3 is a theta function.

Note that the 1- and 2-form heat kernels are derived in the same manner, with the same results, since the sum over the group does not affect the differentials.

This same technique can be applied also to the cylinder, where $\mathbb{Z}$ acts by translation (we'll say in the horizontal direction). The result is (for 0-forms)

$$K_0^{(C)}(\mathbf{x}, \mathbf{y}, t) = T(x_1 - y_1, t)(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x_2 - y_2)^2}{4t}} 1_x \otimes 1_y.$$

Since we are not concerned with the metric on the cylinder, the 1- and 2-form heat kernels are basically the same as above, with the appropriate differentials.

We now consider the following,

$$M = \{\mathbf{x} \in \mathbb{R}^3 | x_3^2 = x_1^2 + x_2^2,\ x_3 \geq 0\},$$

which is a circular cone lying above the x_1x_2-plane. If we parameterize the surface with polar coordinates in the following manner

$$x_1 = \frac{r\cos\theta}{\sqrt{2}}$$

$$x_2 = \frac{r\sin\theta}{\sqrt{2}}$$
$$x_3 = \frac{r}{\sqrt{2}}$$

then we have

$$ds^2 = dr^2 + 2^{-1}r^2 d\theta^2.$$

The reason we chosen a factor of $\frac{1}{\sqrt{2}}$ in the parameterization is so a point (r, θ) on the surface would be a distance r from the vertex of the cone.

This cone, M, gives a nice tiling of $\mathbb{R}^2$, with just a rotation of π about the origin. Using the techniques described above, we can write

$$K_0^{(M)}(\mathbf{r}, \mathbf{s}, t) = (4\pi t)^{-1} \sum_{j=0}^{1} e^{-(4t)^{-1}(r^2+\rho^2-2r\rho\cos(\theta-\psi-\pi j))}.$$

This simplifies to

$$K_0^{(M)} = (2\pi t)^{-1} e^{-(4t)^{-1}(r^2+\rho^2)} \cosh((2t)^{-1} r\rho\cos(\theta-\psi)).$$

We arrive at $K_2^{(M)}$ via the Hodge isomorphism, remembering that we use the polar coordinate metric on $\mathbb{R}^2$ and note the metric on the cone.

The 1-form heat kernel is calculated using the 1-form heat kernel on the plane in polar coordinates. Using the fact that $\sin(\theta+\pi) = -\sin(\theta)$ and $\cos(\theta+\pi) = -\cos(\theta)$, we use the above method to determine

$$K_1^{(M)}(\mathbf{r}, \mathbf{s}, t) = \frac{e^{-\frac{r^2+\rho^2}{4t}}}{2\pi t} \sinh\left(\frac{r\rho\cos(\theta-\psi)}{(2t)}\right) \Omega$$

where

$$\Omega = \cos(\theta-\psi)\left(dr \otimes d\rho + r\rho d\theta \otimes d\psi\right) + \sin(\theta-\psi)\left(rd\theta \otimes d\rho - \rho dr \otimes d\psi\right).$$

4.4 Examples: Hyperbolic Cylinder and Cone

To move to quotients of the hyperbolic plane, we will consider the two example given in Example 4.1.6 (5), the hyperbolic cone and the hyperbolic cylinder. The cone is most easily represented as a quotient in the hyperboloid model, while the cylinder uses the upper half-plane model. Since the heat kernel is given for the hyperboloid model, we will consider the cone first.

As stated in Example 4.1.6 (5), the hyperbolic cone, is the quotient of H^2 by the group $\mathbb{Z}_n$, with the group action given by $k \cdot (r, \theta) = (r, \theta + \frac{2\pi k}{n})$. Following the examples in the previous section, we can write the heat kernel for the hyperbolic cone as

$$K(\mathbf{x}, \mathbf{y}, t) = \sum_{k=0}^{n-1} K_{H^2}(\mathbf{x}, k \cdot \mathbf{y}, t). \tag{4.4.1}$$

For 0-forms, this means that

$$\begin{aligned} K_0\left((r,\theta),(R,\phi),t\right) = \frac{1}{2\pi} \sum_{k=0}^{n-1} \int_0^\infty & \rho e^{-\left(\frac{1}{4}+\rho^2\right)t} \tanh \pi\rho \\ & \times P_{-\frac{1}{2}+i\rho}\left(\cosh d_{H^2}\left((r,\theta),\left(R,\phi+\frac{2\pi k}{n}\right)\right)\right) d\rho. \end{aligned} \tag{4.4.2}$$

By using the Hodge star isomorphism, we are able to write the 2-form heat kernel using the 0-form heat kernel.

In the case of the 1-form heat kernel, we must consider $*_{k\cdot\mathbf{y}}$ and $d_{k\cdot\mathbf{y}}$. However, in each of these cases, the group action does not change the operator, so we can write

$$\begin{aligned} K_1((r,\theta),(R,\phi,t)) = \frac{1}{2\pi} \sum_{k=0}^{n-1} \left(I + *_{(r,\theta)}*_{(R,\phi)}\right) d_{(r,\theta)} d_{(R,\phi)} \int_0^\infty & \rho e^{-\left(\frac{1}{4}+\rho^2\right)t} \tanh \pi\rho \\ & \times P_{-\frac{1}{2}+i\rho}\left(\cosh d_{H^2}\left((r,\theta),\left(R,\phi+\frac{2\pi k}{n}\right)\right)\right) d\rho. \end{aligned} \tag{4.4.3}$$

$$
\begin{tikzcd}
P \arrow[r, "u"] & U \arrow[r, "n"] & U \arrow[r, "u^{-1}"] \arrow[ddr] & P \arrow[d, "r^{-1}"] \\
R \arrow[u, "r"] & & & R \arrow[d, "p^{-1}"] \\
H^2 \arrow[u, "p"] \arrow[uur] \arrow[rrr] & & & H^2
\end{tikzcd}
$$

Figure 4.4.1: Defining the group action for the hyperbolic cylinder.

When we consider the hyperbolic cylinder, we first move from the hyperboloid model to the upper half-plane model, since the group action is easier to express in that model. After we state the group action in the upper half-plane model, we use the relationships given in Table 3.1.2 to write the group action in the hyperboloid model.

Recalling from Example 4.1.6 (5), we consider the hyperbolic cone to the be quotient of the hyperbolic plane, UHP, and the group $\mathbb{Z}$, where the group action is given by $n \cdot (x, y) = (x + n, y)$. This means we can write the heat kernel on the hyperbolic cylinder as

$$K(\mathbf{x}, \mathbf{y}, t) = \sum_{n=-\infty}^{\infty} K^{(H^2)}(\mathbf{x}, n \cdot \mathbf{y}, t). \tag{4.4.4}$$

Because of the complicated expression for the group action in the hyperboloid model, we choose not to expand this equation. However, in Figure 4.4.1, we show the steps for moving the group action from the upper half-plane to the hyperboloid. The maps p, r, and u are the change of coordinates. We use the polar coordinate Poincaré disk, R, as an intermediate step. The map denoting the group action is n, and the unlabeled maps are defined so the diagram commutes. For the map u, we refer back to Table 3.1.2. The map r is the usual change of coordinates from polar to rectangular, and $p(\eta, \theta) = (\tanh(\frac{\eta}{2}), \theta)$.

Chapter 5
Conclusion

We have now been able to demonstrate the computation of the differential forms heat kernel for the hyperbolic plane. In addition, we have presented a method of relating the 0-form heat kernel, or the heat kernel for functions, to the 1-form heat kernel, thereby taking previously known results and extending them with little cost to the differential forms situation.

In addition to computing the heat kernel for the hyperbolic plane, we have introduced the notion of the projecting that heat kernel onto quotients of H^2. This allows us to write the heat kernel of Riemann surfaces in terms of the heat kernel of their universal covering space.

The technique given for computing the heat kernel for quotients of hyperbolic space we have developed may play a role in establishing a characterisation of non-compact Riemann, and psuedo-Riemannian surfaces. This characterisation could take a form similar to the Gauss-Bonnet formula for compact Riemann surfaces.

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