Research Article

A Test Matrix for an Inverse Eigenvalue Problem

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1. Introduction

We are motivated by the following inverse eigenvalue problem first studied by Hochstadt in 1967 [1]. Given two strictly interlaced sequences of real values,

\[(\lambda_i^1)^n, \quad (\lambda_i^o)^{n-1}\]

with

\[\lambda_1^1 < \lambda_1^o < \lambda_2^1 < \lambda_2^o < \cdots < \lambda_{n-1}^o < \lambda_n^o < \lambda_n\]

find the \(n \times n\), real, symmetric, and tridiagonal matrix, \(B\), such that \(\lambda(B) = (\lambda_i^1)_1^n\) are the eigenvalues of \(B\), while \(\lambda(B^o) = (\lambda_i^o)^{n-1}_{1}\) are the eigenvalues of the leading principal submatrix of \(B\), where \(B^o\) is obtained from \(B\) by deleting the last row and column. The condition on the dataset (2) is both necessary and sufficient for the existence of a unique Jacobian matrix solution to the problem (see [2], Section 4.3 or [3], Section 1.2 for a history of the problem and Section 3 of this paper for additional background theory).

A number of different constructive procedures to produce the exact solution of this inverse problem have been developed [4–9], but none provide an explicit characterization of the entries of the solution matrix, \(B\), in terms of the dataset (2). Computer implementation of these procedures introduces floating point error and associated numerical stability issues. Loss of significant figures due to accumulation of round-off error makes some of the known solution procedures undesirable. Determining the extent of round-off error in the numerical solution, \(\hat{B}\), computed from a given dataset requires a priori knowledge of the exact solution \(B\). In the absence of this knowledge, an additional numerical computation of the forward problem to find the spectra \(\lambda(\hat{B})\) and \(\lambda(B^o)\) allows comparison to the original data.

Test matrices, with known entries and known spectra, are therefore helpful in comparing the efficacy of the various solution algorithms in regard to stability. It is particularly helpful when test matrices can be produced at arbitrary size. However some existent test matrices given as a function of matrix size \(n\) suffer the following trait: when ordered by size, the minimum spacing between consecutive eigenvalues is a decreasing function of \(n\). This trait is potentially undesirable since the reciprocal of this minimum separation between eigenvalues can be thought of as a condition number on the sensitivity of the eigenvectors (invariant subspaces) to perturbation (see [10], Theorem 8.1.12). Some of the algorithms for the inverse problem seem to suffer from this form of ill-conditioning. From a motivation to avoid confounding the numerical stability issue with potential increased ill-conditioning of the dataset as a function of \(n\), the authors developed a test matrix which has equally spaced and uniformly interlaced simple eigenvalues.
In Section 2 we provide the explicit entries of such a matrix, $A(n)$. We claim that its eigenvalues are equally spaced as

$$
\lambda(A(n)) = \{0, 2, 4, \ldots, 2n - 2\},
$$

while its leading principal submatrix $A^o(n)$ has eigenvalues uniformly interlaced with those of $A(n)$, namely,

$$
\lambda(A^o(n)) = \{1, 3, 5, \ldots, 2n - 3\}.
$$

A short proof verifies the claims. In Section 3 we present some background theory concerning Jacobian matrices, and in Section 4 we apply our test matrix to a model of a physical spring-mass system, an application which leads naturally to Jacobian matrices.

2. Main Result

Let $A(n)$ be an $n \times n$ real symmetric tridiagonal matrix with entries

$$
a_{ii} = n - 1, \quad i = 1, 2, \ldots, n
$$

$$
a_{i,i+1} = \frac{1}{2} \sqrt{i(2n-i-1)}, \quad i = 1, 2, \ldots, n-2
$$

$$
a_{n-1,n} = \sqrt{\frac{n(n-1)}{2}}
$$

and let $A^o(n)$ be the principal submatrix of $A(n)$, that is, the $(n-1) \times (n-1)$ matrix obtained from $A(n)$ by deleting the last row and column.

**Theorem 1.** $A(n)$ has eigenvalues $\{0, 2, \ldots, 2n - 2\}$ and $A^o(n)$ has eigenvalues $\{1, 3, \ldots, 2n - 3\}$.

**Proof.** By induction, when $n = 2$

$$
A(2) = \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
$$

has eigenvalues 0, 2, and $A^o(2)$ has eigenvalue 1. Assume the result holds for $n$. So $A(n)$ has eigenvalues $\{0, 2, \ldots, 2n - 2\}$. Let $B = A^o(n+1) - nI$ and $A = A(n) - (n-1)I$. Then $B$ and $A$ are similar via $BR = RA$ where $R$ is upper triangular, with entries

$$
r_{ij} = \begin{cases}
\frac{k(j-1)!(2n-j-1)!}{(i-1)!(2n-i+1)!} & i, j \text{ have same parity and } j \geq i, \\
0 & \text{otherwise},
\end{cases}
$$

$$
k = \begin{cases}
2 & j \neq n, \\
1 & j = n.
\end{cases}
$$

Therefore $A^o(n+1)$ has eigenvalues $\{1, 3, \ldots, 2n - 1\}$.

Now we show that $A(n+1)$ has eigenvalues $\{2n\} \cup \{\text{eigenvalues of } A(n)\}$. Let $C = A(n+1) - 2nI$. Factorize $C = -LL^T$, where $L$ is lower bidiagonal. We find

$$
l_{ii} = \sqrt{\frac{2n-i+1}{2}}; \quad l_{i+1,i} = -\sqrt{i}; \quad i = 1, 2, \ldots, n - 1,
$$

$$
l_{nn} = \sqrt{\frac{n+1}{2}}; \quad l_{n+1,n} = -\sqrt{n}; \quad l_{n+1,n+1} = 0.
$$

Therefore $C$ has eigenvalue 0 and thus $A(n+1)$ has eigenvalue $2n$.

Define $D = 2nI - LL^T$; so

$$
D = \begin{bmatrix}
D & O \\
O & 2n
\end{bmatrix}
$$

with

$$
d_{ii} = \frac{2n-1}{2}; \quad d_{i+1,i} = \frac{1}{2} \sqrt{i(2n-i)}, \quad i = 1, 2, \ldots, n - 1,
$$

$$
d_{nn} = \frac{n-1}{2}.
$$

Now $D^o$ has the same eigenvalues as $A(n)$ since they are similar matrices via $SD^o = A(n)S$ where $S$ is upper triangular with entries

$$
s_{ii} = \sqrt{2n-i}; \quad s_{i+1,i} = -\sqrt{i}, \quad i = 1, 2, \ldots, n - 1,
$$

$$
s_{nn} = \sqrt{2n}; \quad s_{ij} = 0, \quad \text{otherwise}.
$$

Therefore $A(n+1)$ has eigenvalues $\{2n\} \cup \{\text{eigenvalues of } A(n)\}$. $\square$

3. Discussion

A real, symmetric $n \times n$ tridiagonal matrix $B$ is called a Jacobian matrix when its off-diagonal elements are nonzero ([2], page 46). We write

$$
B = \begin{bmatrix}
a_{11} & -b_1 & 0 & 0 & \cdots & 0 \\
-b_1 & a_{22} & -b_2 & 0 & \cdots & 0 \\
0 & -b_2 & a_{33} & -b_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-1, n-1} & -b_{n-1} \\
0 & 0 & 0 & \cdots & 0 & a_{nn}
\end{bmatrix}
$$

The similarity transformation, $\tilde{B} = S^{-1}BS$, where $S = S^{-1}$ is the alternating sign matrix, $S = \text{diag}(1, -1, 1, -1, \ldots, (-1)^{n-1})$, produces a Jacobian matrix $\tilde{B}$ with entries same as $B$ except for the sign of the off-diagonal elements, which are all reversed. If instead we use the self-inverse sign matrix, $S^{(m)} = \text{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1)$, to transform $B$, then $\tilde{B}$ is a Jacobian matrix identical to $B$ except for a switched sign on the $m$th off-diagonal element. In regard to the spectrum of
the matrix, there is therefore no loss of generality in accepting
the convention that a Jacobian matrix is expressed with
negative off-diagonal elements; that is, \( b_i > 0 \), for all \( i = 1, \ldots, n-1 \) in (13).

While Cauchy’s interlace theorem [11] guarantees that
the eigenvalues of any square, real, symmetric (or even
Hermitian) matrix will interlace those of its leading (or
trailing) principal submatrix, the interlacing cannot be strict,
in general [12]. However, specializing to the case of Jacobian
matrices restricts the interlacing to strict inequalities. That
is, Jacobian matrices possess distinct eigenvalues, and the
eigenvalues of the leading (or trailing) principal submatrix
are also distinct and strictly interlace those of the original
matrix (see [2], Theorems 3.1.3 and 3.1.4. See also [10]
exercise P8.4.1, page 475: when a tridiagonal matrix has
algebraically multiple eigenvalues, the matrix fails to be
Jacobian). The inverse problem is also well-posed: there is
a unique (up to the signs of the off-diagonal elements)
Jacobian matrix \( B \) having given spectra specified as per (2)
(see [2], Theorem 4.2.1, noting that the interlaced spectrum
of \( n-1 \) eigenvalues \( \lambda^{(n-1)} \) can be used to calculate the last
components of each of the \( n \) orthonormalized eigenvectors
of \( B \) via equation 4.3.11). Therefore, the matrix \( A(n) \) in
Theorem 1 is the unique Jacobian matrix with eigenvalues
equally spaced by two, starting with smallest eigenvalue
zero, whose leading principal submatrix has eigenvalues also
equally spaced by two, starting with smallest eigenvalue one.

As a consequence of the theorem, we now have the
following.

**Corollary 2.** The eigenvalues of the real, symmetric \( n \times n \)
tridiagonal matrix

\[
W_n = \begin{bmatrix}
\cdots & 0 & 0 \\
0 & -c \sqrt{\frac{n-1}{2}} & a \\
-\sqrt{\frac{n-1}{2}} & a & -c \sqrt{\frac{2n-3}{2}} \\
0 & 0 & \ddots \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & -c \sqrt{\frac{n(n-1)}{2}} & a
\end{bmatrix}
\]

form the arithmetic sequence,

\[
\lambda(W_n) = \{a_o + 2c(i-1)\}_{i=1}^n, \quad (15)
\]

while the eigenvalues of its leading principal submatrix, \( W_n^o \),
form the uniformly interlaced sequence

\[
\lambda(W_n^o) = \{a_o + c + 2c(i-1)\}_{i=1}^{n-1}, \quad (16)
\]

where \( a = a_o + c(n-1) \).

The form and properties of \( W_n \) were first hypothesised
by the third author while programming Fortran algorithms
to reconstruct band matrices from spectral data [3]. Initial
attempts to prove the spectral properties of \( W_n \) by both he
and his graduate supervisor (the first author) failed. Later, the first
author produced the short induction argument of Theorem 1,
in July 1996. Alas, the fax on which the argument was
communicated to the third author was lost in a cross-border
academic move, and so the matter languished until recently.
In summer of 2013, the second and third authors assigned the
problem of this paper as a summer undergraduate research
project, “hypothesize, and then verify, if possible, the explicit
entries of an \( n \times n \) symmetric, tridiagonal matrix with
eigenvalues (15), such that the eigenvalues of its principal
submatrix are (16).” Meanwhile the misplaced fax from the
first author’s proof was found during an office cleaning. The
student, A. De Serre-Rothney, was able to complete both parts
of the problem. His proof is now found in [13]. Though longer
than the one presented here, his proof utilizes the spectral
properties of another tridiagonal (nonsymmetric) matrix, the
so-called Kac-Sylvestermatrix, \( K_n \), of size \((n+1) \times (n+1)\), with
eigenvalues \( \lambda(K_n) = \{2k-n\}_{k=0}^n \) [14–17]:

\[
K_n = \begin{bmatrix}
n & n-1 & 0 & 0 & \cdots & 0 \\
n-1 & n-2 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \cdots & \cdots \\
\vdots & \vdots & \ddots & n-2 & 0 & n \\
0 & 0 & \cdots & \cdots & 0 & n-1 \\
0 & 0 & \cdots & \cdots & \cdots & n
\end{bmatrix} \quad (17)
\]

The referee has pointed out the connection between
the spectra (3) and (4) and the classical orthogonal Hahn
polynomials of a discrete variable [18]. Using (3) as nodes
with weights

\[
\omega_i = \frac{\prod_{j=1}^{i-1} (\lambda_i - \lambda_j^2)}{\prod_{j=1}^{n} (\lambda_i - \lambda_j)}, \quad i = 1, \ldots, n, \quad (18)
\]

determine the Hahn polynomials, \( h_k^{1/2,1/2}(x/2, n) \), \( k = 0, 1, \ldots, n-1 \), whose three-term recurrence coefficients are
the entries of a Jacobi matrix with eigenvalues (3), hence
similar to our \( A(n) \).
4. A Spring-Mass Model Problem

One simple problem where symmetric tridiagonal matrices arise naturally is the inverse problem for the spring-mass system shown in Figure 1. In this case the squares of the natural frequencies of free vibration for system (a) are the eigenvalues of a Jacobi matrix $B$, while those for system (b) are the eigenvalues of its principal minor $B^o$.

Specifically, let $C$ be the stiffness matrix, and let $M$ be the mass (inertia) matrix for the system in Figure 1(a):

$$C = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \ldots & -k_2 \\ -k_2 & k_2 + k_3 & -k_3 & & & \ldots & -k_3 \\ & -k_3 & k_{n-1} + k_n & -k_n & & \ldots & -k_n \\ & & -k_n & & & \ldots & -k_n \\ & & & & k_1 + k_2 & \ldots & -k_2 \\ & & & & -k_2 & \ldots & k_1 + k_2 \end{bmatrix},$$

$$M = \begin{bmatrix} m_1 & & & & & & \\ & m_2 & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & m_n \end{bmatrix}.$$

Then the squares of the natural frequencies of the systems in Figure 1 satisfy $(C - \lambda M)x = 0$ and $(C^o - \lambda^o M^o)x^o = 0$, where $C'$ is obtained from $C$ by deleting the last row and column. The solutions can be ordered $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n$. We can also rewrite the systems as $(B - \lambda I)u = 0$ and $(B^o - \lambda^o I)u^o = 0$ where $B = M^{-1/2}CM^{-1/2}$ and $u = M^{1/2}x$. Note that the squares of the natural frequencies of the systems are the eigenvalues of $B$ and $B^o$.

Suppose that the matrix $B(n) := A(n) + I$ was to arise from a spring-mass system like in Figure 1; that is, we are considering the system whose squares of the natural frequencies are the equally spaced values $\{1, 3, \ldots, 2n - 1\}$ for system (a) and $\{2, 4, \ldots, 2n - 2\}$ for system (b). The system in Figure 1 is the simplest possible discrete model for a rod vibrating in longitudinal motion and more closely approximates the continuous system as $n \to \infty$. In a physical system, we expect clustering of frequencies. The test matrix $B(n)$ does not share this phenomenon and so we expect the stiffnesses and masses associated with $B(n)$.

With $B(n) = A(n) + I$ we note that

$$B_{ii} = a_i = n, \quad i = 1, \ldots, n$$

$$B_{i,i+1} = -b_i = -\frac{1}{2} \sqrt{i(2n - i - 1)}, \quad i = 1, \ldots, n - 2$$

$$B_{n-1,n} = -b_{n-1} = -\frac{n(n - 1)}{2}$$

with eigenvalues $\{2k + 1\}_{k=0}^{n-1}$, while $B^o(n)$ has eigenvalues $\{2k\}_{k=1}^{n-1}$.

Let $u = \langle m_1^{1/2}, \ldots, m_n^{1/2} \rangle^T$ with $m_i > 0$ for all $i$. Let $m = \sum_{i=1}^{n} m_i = u^T u$. We wish to solve

$$B(n) u = \langle m_1^{1/2}, m_2^{1/2}, \ldots, m_n^{1/2} \rangle^T$$

for $(m_i)_{i=1}^{n}$ and $k_1$.

The bottom, $n$th, equation is

$$m_{n-1}^{1/2} = \frac{-nm_{n-2}^{1/2}}{-b_{n-2}} = \sqrt{2} \left( \frac{n}{n-1} \right)^{1/2} \alpha,$$

where we choose $m_n^{1/2} = \alpha$. We will thus be able to express $m_i^{1/2}$ in terms of the scaling parameter $\alpha$.

The $(n - 1)$th equation is

$$m_{n-2}^{1/2} = \frac{\alpha b_{n-1} - nm_{n-2}^{1/2}}{-b_{n-2}} = \alpha \frac{\sqrt{n(n-1)/2} - n \sqrt{2} \sqrt{n/(n-1)}}{-(1/2) \sqrt{n-2} (n+1)}$$

$$= \alpha \sqrt{2} \left( \frac{(n+1)}{(n-1)(n-2)} \right)^{1/2}.$$  \hspace{1cm} (23)

The $i$th equation, for $i \neq 1, n - 1, n$, is

$$-b_{i-1}m_i^{1/2} + nm_i^{1/2} - b_im_{i+1}^{1/2} = 0.$$  \hspace{1cm} (24)

Then

$$m_{i+1}^{1/2} = \frac{2nm_i^{1/2} - (i(2n - i - 1))^{1/2}m_{i+1}^{1/2}}{(i - 1)(2n - i))^{1/2}}.$$  \hspace{1cm} (25)

Now suppose

$$m_{n-i}^{1/2} = \alpha \sqrt{2} \left( \frac{n(n+1) \cdots (n+i-1)}{(n-1)(n-2) \cdots (n-i)} \right)^{1/2}.$$  \hspace{1cm} (26)
for $i = 1, 2, \ldots, j$. Then cases $i = 1, 2$ are already verified, and
the strong inductive assumption applied in (25) with $i - 1 = n - (j + 1)$
implies $i = n - j$. So

$$m_{n-j-1} = \left(2n\alpha\sqrt{\frac{(n(n+1)\cdots(n+j-1)}{((n-1)(n-2)\cdots(n-j))}}\right)^{1/2} \times \left((-n-j)(n+j-1)^{1/2}ight) \times \left(\alpha\sqrt{\frac{(n(n+1)\cdots(n+j-1)}{(n-1)(n-2)\cdots(n-j))}\right)^{1/2} \times \left(\frac{2n}{n-j}\right)

$$

which verifies, by strong induction, the closed form for $m_{n-j}$
given by (26).

Finally, the first equation of (21) is

$$nm_1^{1/2} - b_1 m_2^{1/2} = m_1^{1/2} k_1

$$

and so

$$k_1 = nm_1 - b_1 (m_1 m_2)^{1/2}.

$$

We note that the values $m_{n-i}$ can be written as

$$m_{n-i} = 2\alpha^2 \frac{(n+i-1)! (n-i)!}{((n-1)!)^2}

$$

for $i = 1, \ldots, n-1$, and

$$m_n = \alpha^2 \frac{(n+0-1)! (n-0-1)!}{((n-1)!)^2} = \alpha^2.

$$

Since $C = M^{1/2} B(n)M^{1/2}$, then

$$k_{i+1} = -C_{i+1} = -m_{n-(n-i)}^{1/2} B_{i+1} m_{n-(n-i-1)}^{1/2} = \alpha^2 \frac{i! (2n-i-1)!}{((n-1)!)^2},

$$

(32)

$$k_1 = \alpha^2 \frac{(2n-1)!}{((n-1)!)^2}.

$$

(33)

From (26) we have $m_1/m_n = 2((2n-2)!/((n-1)!)^2)$ which goes to infinity as $n \to \infty$ and from (32) we see that $k_1/k_n = (2n-1)!/(n-1)!n!$ which also goes to infinity as $n \to \infty$.

This is not a model of a physical rod, as expected.

5. Conclusion

A family of $n \times n$ symmetric tridiagonal matrices, $W_n$, whose
eigenvalues are simple and uniformly spaced and whose
leading principle submatrix has uniformly interlaced, simple
eigenvalues has been presented (14). Members of the family
are characterized by a specified smallest eigenvalue $a_0$ and gap
size $c$ between eigenvalues. The matrices are termed Jacobian,
since the off-diagonal entries are all nonzero. The matrix
entries are explicit functions of the size $n$, $a_0$, and $c$; so the
matrices can be used as a test matrices for eigenproblems,
both forward and inverse. The matrix $W_n$ for specified
smallest eigenvalue $a_0$ and gap $c$ is unique up to the signs of
the off-diagonal elements.

In Section 4, the form of $W_n$ was used as an explicit
solution of a spring-mass vibration model (Figure 1), and
the inverse problem to determine the lumped masses and
spring stiffnesses was solved explicitly. Both the lumped
masses $m_{n-i}$ given by (30) and spring stiffnesses $k_{n-i}$ from (32)
show superexponential growth. Consequently $m_n/m_1$, $k_n/k_1$
become vanishingly small as $n \to \infty$. As a result, the spring-
mass system of Figure 1 cannot be used as a discretized model
for a physical rod in longitudinal vibration, as the model
becomes unrealistic in the limit as $n \to \infty$.

Conflict of Interests

The authors declare that there is no conflict of interests
regarding the publication of this paper.

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