Heat Kernel for Open Manifolds

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22[nd] July, 2010

Abstract

It is known that for open manifolds with bounded geometry, the differential form heat kernel exists and is unique. Furthermore, it has been shown that the components of the differential form heat kernel are related via the exterior derivative and the coderivative. We will give a proof of this condition for complete manifolds with Ricci curvature bounded below, and then use it to give an integral representation of the heat kernel of degree k.

Keywords: heat kernel, differential forms, Green's functions
MSC: 58J35, 35K08

Publication Informaion: Differential Geometry and its Applications, 28 (2010), pp. 518-522

1 Introduction

In this paper we are considering the differential forms heat equation on manifolds, in particular we are considering $(\Delta + \partial_t)\omega = 0$ with Dirichlet initial conditions. Our goal is to produce a formula for the Green's function, also known as the heat kernel or fundamental solution, which gives the solution of this equation.

The solutions of this equation in the case of functions, or 0-forms, is well-known. The work on differential forms has been much more recent. In 1983, Dodziuk [4] proved that for complete oriented C^∞ Riemannian manifolds with Ricci curvature bounded below, bounded solutions are uniquely determined by their initial values. In a 1988 paper by Buttig, [1], the author listed in Appendix A.2 properties of a "good heat kernel". In 1991, Buttig and Eichhorn [2] were able to give an existence and uniqueness proof for Buttig's conjecture for the differential forms heat kernel on open manifolds of bounded geometry. One of the properties given by Buttig and Eichhorn for a global heat kernel was that the heat kernels $K_k(\mathbf{x}, \mathbf{y}, t)$ and $K_{k+1}(\mathbf{x}, \mathbf{y}, t)$ are related by

$$d_{\mathbf{x}} K_k(\mathbf{x}, \mathbf{y}, t) = d^*_{\mathbf{y}} K_{k+1}(\mathbf{x}, \mathbf{y}, t). \tag{1.1}$$

Here, K_k refers to the degree k portion of the heat kernel. We will use the terminology "k-form heat kernel" to refer to the degree k component of the heat kernel. Using that identity (1.1), we have previously shown, [5], that the 1-form heat kernel on open Riemann surfaces of bounded geometry has the form

$$K_1(\mathbf{x}, \mathbf{y}, t) = (I + *_{\mathbf{x}} *_{\mathbf{y}})\, d_{\mathbf{x}} d_{\mathbf{y}} \int_t^\infty K_0(\mathbf{x}, \mathbf{y}, \tau) d\tau.$$

This directly relates the 1-form heat kernel to the 0-form heat kernel, about which more is known.

In this article, we will present a proof of this property for manifolds with Ricci curvature bounded below, and then use this to give a formula for the k-form heat kernel.

2 Ricci curvature bounded below

To start, we will established the identity (1.1) for complete manifolds with Ricci curvature bounded below. We use the condition on the Ricci curvature to guarantee the existence and uniqueness of the differential forms heat kernel.

Lemma 2.1 *For a complete manifold M, with Ricci curvature bounded from below, we have the following relationship between the k- and $(k+1)$-form heat kernels:*

1. $d_{\mathbf{x}} K_k(\mathbf{x}, \mathbf{y}, t) = d^*_{\mathbf{y}} K_{k+1}(\mathbf{x}, \mathbf{y}, t)$
2. $d_{\mathbf{y}} K_k(\mathbf{x}, \mathbf{y}, t) = d^*_{\mathbf{x}} K_{k+1}(\mathbf{x}, \mathbf{y}, t)$

Proof: Let $E(\mathbf{x}, \mathbf{y}, t) = d_{\mathbf{x}} K_k(\mathbf{x}, \mathbf{y}, t) - d^*_{\mathbf{y}} K_{k+1}(\mathbf{x}, \mathbf{y}, t)$. We will demonstrate that E satisfies the heat equation with zero as the initial condition. This will imply, by uniqueness of the solutions of the heat equation, see [4], that $E \equiv 0$, giving the desired result.

First

$$\begin{aligned} \Delta_{\mathbf{x}} E &= \Delta_{\mathbf{x}} d_{\mathbf{x}} K_k(\mathbf{x}, \mathbf{y}, t) - \Delta_{\mathbf{x}} d^*_{\mathbf{y}} K_{k+1}(\mathbf{x}, \mathbf{y}, t) \\ &= d_{\mathbf{x}} \Delta_{\mathbf{x}} K_k(\mathbf{x}, \mathbf{y}, t) - d^*_{\mathbf{y}} \Delta_{\mathbf{x}} K_{k+1}(\mathbf{x}, \mathbf{y}, t) \\ &= d_{\mathbf{x}}(-\partial_t) K_k(\mathbf{x}, \mathbf{y}, t) - d^*_{\mathbf{y}}(-\partial_t) K_{k+1}(\mathbf{x}, \mathbf{y}, t) \\ &= -\partial_t E. \end{aligned}$$

Next consider $W := \langle E, \omega(\mathbf{x}) \rangle$, where ω is a suitable test function and $\langle \mu, \nu \rangle = \int_M \mu \wedge *\nu$. Then

$$\begin{aligned} \lim_{t \to 0+} W &= \lim_{t \to 0+} \langle d_{\mathbf{x}} K_k, \omega(\mathbf{x}) \rangle - \left\langle d^*_{\mathbf{y}} K_{k+1}, \omega(\mathbf{x}) \right\rangle \\ &= \lim_{t \to 0+} \langle K_k, d^*_{\mathbf{x}} \omega(\mathbf{x}) \rangle - d^*_{\mathbf{y}} \langle K_{k+1}, \omega(\mathbf{x}) \rangle \\ &= d^*_{\mathbf{y}} \omega(\mathbf{y}) - d^*_{\mathbf{y}} \omega(\mathbf{y}) = 0 \end{aligned}$$

Since ω was an arbitrary test function, we must have that $E \equiv 0$ at $t = 0$. Thus by uniqueness, $E \equiv 0$ for all $t > 0$.

The proof of the second assertion follows in a similar manner. □

We will use this result to give an explicit formula for K_k in terms of $K_{k \pm 1}$.

Theorem 2.2 *Let M be an open, complete manifold with Ricci curvature bounded below. Then the differential forms heat kernel obey the following relation:*

$$K_k(\mathbf{x}, \mathbf{y}, t) = d_{\mathbf{x}} d_{\mathbf{y}} \int_t^\infty K_{k-1}(\mathbf{x}, \mathbf{y}, \tau) d\tau + d^*_{\mathbf{x}} d^*_{\mathbf{y}} \int_t^\infty K_{k+1}(\mathbf{x}, \mathbf{y}, \tau) d\tau.$$

Proof: Let K_k be the k-form heat kernel. Clearly,

$$K_k(\mathbf{x}, \mathbf{y}, t) = -\int_t^\infty \frac{\partial}{\partial \tau} K_k(\mathbf{x}, \mathbf{y}, \tau) d\tau,$$

since K_k tends to zero (pointwise) as t increases. Since K_k is a solution of the heat equation, we can replace the time derivative with $-\Delta_{\mathbf{x}} = -d_{\mathbf{x}} d^*_{\mathbf{x}} - d^*_{\mathbf{x}} d_{\mathbf{x}}$, so

$$K_k(\mathbf{x}, \mathbf{y}, t) = \int_t^\infty (d_{\mathbf{x}} d^*_{\mathbf{x}} + d^*_{\mathbf{x}} d_{\mathbf{x}}) K_k(\mathbf{x}, \mathbf{y}, \tau) d\tau.$$

Using Lemma 2.1, we can rewrite the above as

$$K_k(\mathbf{x}, \mathbf{y}, t) = \int_t^\infty d_{\mathbf{x}} d_{\mathbf{y}} K_{k-1}(\mathbf{x}, \mathbf{y}, \tau) + d^*_{\mathbf{x}} d^*_{\mathbf{y}} K_{k+1}(\mathbf{x}, \mathbf{y}, \tau) d\tau.$$

□

The result in Theorem 2.2 depends mainly on two things: the existence and uniqueness of the heat kernel, and the pointwise convergence to zero of the kernel for large time. The methods used above work for a diffusion-type equation provided these conditions are met. For example, for the diffusion equation $(\Delta + c\partial_t)\omega = 0$, the proofs follow through almost identically.

Corollary 2.3 *Let M be an open, n-dimensional, differentiable manifold, with Ricci curvature bounded below, and consider the differential form diffusion equation $(\Delta + c\partial_t)\omega = 0$ with initial data $\omega(\mathbf{x}, 0) = f(\mathbf{x})$. Then the Green's functions are related by*

$$G_k(\mathbf{x}, \mathbf{y}, t) = d_{\mathbf{x}} d_{\mathbf{y}} \int_{ct}^\infty G_{k-1}(\mathbf{x}, \mathbf{y}, \tau) d\tau + d^*_{\mathbf{x}} d^*_{\mathbf{y}} \int_{ct}^\infty G_{k+1}(\mathbf{x}, \mathbf{y}, \tau) d\tau.$$

Proof: Let $T = ct$, then $c\partial_t = \partial_T$, so the equation becomes $(\Delta + \partial_T)\,\omega(\mathbf{x}, T) = 0$ with the same initial conditions. So by Theorem 2.2 we have the desired Green's functions. □

3 Two-dimensional manifolds

In the case of 2-dimensional manifolds, the 0-form and the 2-form heat kernels are isomorphic, as the following Lemma will show. This allows us to write the 1-form heat kernel in terms of the 0-form, or function, heat kernel.

Lemma 3.1 *Let M be a complete manifold with Ricci curvature bounded below. Then the differential forms heat kernels, K_k and K_{n-k} are related in the following manner:*

$$K_k = *_{\mathbf{x}} *_{\mathbf{y}} K_{n-k}.$$

Proof: Consider the equation $(\partial_t + \Delta_k)u = 0,\;\; u(\mathbf{x}, 0) = f(\mathbf{x})$. Then u is given by

$$u(\mathbf{x}, t) = \langle K_k(\mathbf{x}, \mathbf{y}, t), f(\mathbf{y}) \rangle = \int_M K(\mathbf{x}, \mathbf{y}, t) \wedge_{\mathbf{y}} *_{\mathbf{y}} f(\mathbf{y}).$$

Since $*\Delta_k = \Delta_{n-k}*$, it follows that $*_{\mathbf{x}}u$ is a solution of the $(n-k)$-form heat equation with initial condition $*_{\mathbf{x}}f(\mathbf{x})$. So

$$\begin{aligned} *_{\mathbf{x}}u(\mathbf{x},t) &= \langle K_{n-k}(\mathbf{x},\mathbf{y},t), *_{\mathbf{y}}f(\mathbf{y})\rangle \\ &= \langle *_{\mathbf{y}}f(\mathbf{y}), K_{n-k}(\mathbf{x},\mathbf{y},t)\rangle \\ &= \int_M *_{\mathbf{y}}f(\mathbf{y}) \wedge_{\mathbf{y}} *_{\mathbf{y}}K_{n-k}(\mathbf{x},\mathbf{y},t) \end{aligned}$$

By applying $*_{\mathbf{x}}$ to both sides, and changing order in the wedge product, we have

$$(-1)^{k(n-k)}u(\mathbf{x},t) = \int_M (-1)^{k(n-k)} *_{\mathbf{x}} *_{\mathbf{y}} K_{n-k}(\mathbf{x},\mathbf{y},t) \wedge_{\mathbf{y}} *_{\mathbf{y}}f(\mathbf{y})$$

or

$$u(\mathbf{x},t) = \langle *_{\mathbf{x}} *_{\mathbf{y}} K_{n-k}(\mathbf{x},\mathbf{y},t), f(\mathbf{y})\rangle .$$

By uniqueness of the heat kernel we have the desired result. □

Corollary 3.2 *Let M be an open, complete manifold of dimension 2 with Ricci curvature bounded below. Then the 1-form heat kernel on M is given by*

$$K_1(\mathbf{x},\mathbf{y},t) = (I + *_{\mathbf{x}}*_{\mathbf{y}})\, d_{\mathbf{x}}d_{\mathbf{y}} \int_t^\infty K_0(\mathbf{x},\mathbf{y},\tau)d\tau$$

where, $\mathbf{x},\mathbf{y} \in M$ and $t > 0$ and K_0 is the 0-form heat kernel.

Proof: Since M has dimension 2, and so $K_2 = *_{\mathbf{x}} *_{\mathbf{y}} K_0$ by Lemma 3.1. Recall that $d^* * \omega = - * d\omega$ for 0-forms, ω. This gives the desired result. □

As an example, consider the case of the hyperbolic plane, with constant curvature -1. From [3] we have the 0-form heat kernel

$$K_0(\mathbf{x},\mathbf{y},t) = \frac{1}{2\pi}\int_0^\infty P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{x},\mathbf{y}))\rho \exp\left(-\left(\frac{1}{4}+\rho^2\right)t\right)\tanh \pi\rho d\rho,$$

which, if we perform the integration set out in Corollary 3.2, we get

$$K_1(\mathbf{x},\mathbf{y},t) = \frac{1}{2\pi}\,(I + *_{\mathbf{x}}*_{\mathbf{y}})\, d_{\mathbf{x}}d_{\mathbf{y}} \left[\int_0^\infty P_{-\frac{1}{2}+i\rho}(\cosh d_{H^2}(\mathbf{x},\mathbf{y}))\rho \frac{\exp\left(-\left(\frac{1}{4}+\rho^2\right)t\right)}{\frac{1}{4}+\rho^2}\tanh \pi\rho d\rho\right].$$

If M is an open 2-dimensional manifold which has a unique heat kernel for functions, K_0, then Corollary 3.2 suggests a candidate for a heat kernel on 1-forms, and since the K_0 and K_2 heat kernels are isomorphic, we would know all the heat kernels. We will now show that $K_1(\mathbf{x},\mathbf{y},t) = (I + *_{\mathbf{x}}*_{\mathbf{y}})d_{\mathbf{x}}d_{\mathbf{y}}\int_t^\infty K_0(\mathbf{x},\mathbf{y},\tau)d\tau$ works as the heat kernel. Given

$$(\Delta^{(1)}_{\mathbf{x}} + \partial_t)w_1(\mathbf{x},t) = 0 \tag{3.2}$$

$$w_1(\mathbf{x},0) = f_1(\mathbf{x}) \tag{3.3}$$

we will show that w_1 can be written as $w_1(\mathbf{x},t) = \langle K_1(\mathbf{x},\mathbf{y},t), f_1(\mathbf{y})\rangle$.

Let w_1 be a solution of (3.2) and (3.3), and $w(\mathbf{x},t) = \langle K_1(\mathbf{x},\mathbf{y},t), f_1(\mathbf{y})\rangle$. Since the Laplacian commutes with the Hodge star isomorphism and the exterior derivative and coderivative, it is clear that w satisfies equation (3.2). Now we just need to show that w as defined, satisfies the initial condition (3.3).

$$\begin{aligned}
w(\mathbf{x},t) &= \int_t^\infty d_{\mathbf{x}} \langle d_{\mathbf{y}} K_0(\mathbf{x},\mathbf{y},\tau), f_1(\mathbf{y})\rangle + *_{\mathbf{x}} d_{\mathbf{x}} \langle *_{\mathbf{y}} d_{\mathbf{y}} K_0(\mathbf{x},\mathbf{y},\tau), f_1(\mathbf{y})\rangle \, d\tau \\
&= \int_t^\infty d_{\mathbf{x}} \left\langle K_0(\mathbf{x},\mathbf{y},\tau), d^*_{\mathbf{y}} f_1(\mathbf{y})\right\rangle - *_{\mathbf{x}} d_{\mathbf{x}} \langle K_0(\mathbf{x},\mathbf{y},\tau), *_{\mathbf{y}} d_{\mathbf{y}} f_1(\mathbf{y})\rangle \, d\tau \\
&= \int_t^\infty d_{\mathbf{x}} d^*_{\mathbf{x}} w_1(\mathbf{x},\tau) - *_{\mathbf{x}} d_{\mathbf{x}} *_{\mathbf{x}} d_{\mathbf{x}} w_1(\mathbf{x},\tau) d\tau \\
&= \int_t^\infty \Delta w_1(\mathbf{x},\tau) d\tau \\
&= \int_t^\infty -\partial_\tau w_1(\mathbf{x},\tau) d\tau = w_1(\mathbf{x},t)
\end{aligned}$$

Since w_1 is a solution of the heat equation with initial value f_1, and $w = w_1$, this means that w also has initial value f_1. Thus w is a solution of (3.2) and (3.3).

Finally, let us consider the case of compact complete manifolds. In this case, because of conservation, diffusion does not tend to zero, so the large-time limit has to be taken into account.

Theorem 3.3 *Let M be a complete manifold with Ricci curvature bounded below, and let the $\lim_{t\to\infty} K_k(\mathbf{x},\mathbf{y},t)$ be a constant double-form, call it C. Then, the heat kernel obeys the following relation:*

$$K_k(\mathbf{x},\mathbf{y},t) = C + d_{\mathbf{x}} d_{\mathbf{y}} \int_t^\infty K_{k-1}(\mathbf{x},\mathbf{y},\tau) d\tau + d^*_{\mathbf{x}} d^*_{\mathbf{y}} \int_t^\infty K_{k+1}(\mathbf{x},\mathbf{y},\tau) d\tau.$$

Proof: Let K_k be the k-form heat kernel. Clearly,

$$K_k(\mathbf{x},\mathbf{y},t) = C - \int_t^\infty \frac{\partial}{\partial\tau} K_k(\mathbf{x},\mathbf{y},\tau) d\tau,$$

since K_k tends to C as t increases. Since K_k is a solution of the heat equation, we can replace the time derivative with $-\Delta_{\mathbf{x}} = -d_{\mathbf{x}} d^*_{\mathbf{x}} - d^*_{\mathbf{x}} d_{\mathbf{x}}$, so

$$K_k(\mathbf{x},\mathbf{y},t) = C + \int_t^\infty (d_{\mathbf{x}} d^*_{\mathbf{x}} + d^*_{\mathbf{x}} d_{\mathbf{x}}) K_k(\mathbf{x},\mathbf{y},\tau) d\tau.$$

Using Lemma 2.1, we can rewrite the above as

$$K_k(\mathbf{x},\mathbf{y},t) = C + \int_t^\infty \left(d_{\mathbf{x}} d_{\mathbf{y}} K_{k-1}(\mathbf{x},\mathbf{y},\tau) + d^*_{\mathbf{x}} d^*_{\mathbf{y}} K_{k+1}(\mathbf{x},\mathbf{y},\tau)\right) d\tau.$$

□

In conclusion, given an open, complete manifold with Ricci curvature bounded below, the components of the differential form heat kernel are related as follows:

$$K_k(\mathbf{x},\mathbf{y},t) = d_{\mathbf{x}} d_{\mathbf{y}} \int_t^\infty K_{k-1}(\mathbf{x},\mathbf{y},\tau)d\tau + d^*_{\mathbf{x}} d^*_{\mathbf{y}} \int_t^\infty K_{k+1}(\mathbf{x},\mathbf{y},\tau)d\tau.$$

Also, if the manifold is two-dimensional, the components of the heat kernel all depend on the 0-form heat kernel, K_0, with $K_2 = *_{\mathbf{x}} *_{\mathbf{y}} K_0$ and

$$K_1(\mathbf{x},\mathbf{y},t) = (I + *_{\mathbf{x}} *_{\mathbf{y}})\, d_{\mathbf{x}} d_{\mathbf{y}} \int_t^\infty K_0(\mathbf{x},\mathbf{y},\tau)d\tau.$$

Acknowledgements

I would like to thank Daniel Kucerovsky, Sanjeev Seahra, and Kari Dunfield for their helpful comments on earlier versions of this paper.

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