CS 316: Dealing with uncertainty

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THE WEAKEST LINK

- Resolution or modus ponens are exact
  - There is no possibility of mistake if the rules are followed exactly
- These methods of inference (also known as deductive methods) require that information be complete, precise, and consistent
- By contrast, the real world requires common sense reasoning in the face of incomplete, inexact, and potentially inconsistent information

INCOMPLETE FACTS

- A logic is monotonic if the truth of a sentence does not change when more facts are added
  - FOL is for example monotonic
- A logic is non-monotonic if the truth of a proposition may change when new information (facts) is added or old information is deleted
  "It rained last night if the grass is wet and the sprinkler was not on last evening. I am looking right now and see that the grass is wet."
  Did it rain last night?

`rained :- grass_is_wet, \+ sprinkler_was_on.`
`grass_is_wet.`
	`?- rained.`
	`Yes`
	`?- assert(sprinkler_was_on).`
	`Yes`
	`?- rained.`
	`No`
	`?- retract(sprinkler_was_on).`
	`Yes`
	`?- rained.`
	`Yes`

- Similar to the closed world assumption but more precise
- We specify particular predicates that are "as false as possible"
  - Meaning that they are false for all the objects except for those for which we know them to be true

    $$Bird(x) \wedge \neg Abnormal_1(x) \Rightarrow Flies(x)$$

    provided that $Abnormal_1$ is circumscribed
  - We draw the conclusion that $Flies(Tweety)$ out of $Bird(Tweety)$ provided that we do not know that $Abnormal_1(Tweety)$ holds
- Implemented in Prolog by the `not` predicate (more or less)

- Default logic adds a new inference rule: if α is true and β is not known to be false then γ:

$$\frac{\alpha \quad : \quad \beta}{\gamma}$$

e.g.,

$$\frac{\texttt{grass\_is\_wet} \quad : \quad \neg\texttt{sprinkler\_was\_on}}{\texttt{rained}}$$

- Nonmonotonic logic adds a new operator $\mathbb{M}$:

$$\alpha \wedge \mathbb{M}\beta \Rightarrow \gamma$$

stands for "if α is true and β is not known to be false then γ." e.g.,

$$\texttt{grass\_is\_wet} \wedge \mathbb{M}\neg\texttt{sprinkler\_was\_on} \Rightarrow \texttt{rained}$$

$$\begin{array}{c} \mathit{american}(X) \wedge \mathit{adult}(X) \wedge \\ \mathbb{M}(\exists A\ \ (\mathit{car}(A) \wedge \mathit{owns}(X, A))) \end{array} \quad \Rightarrow \quad (\exists A\ \ (\mathit{car}(A) \wedge \mathit{owns}(X, A)))$$

Truth Maintenance Systems

- Problem: If we assert $\neg P$ we will have to retract P (if present)
  - Simple enough, but what if we inferred things starting from P? They will all need to be retracted
  - This retractions are managed by a truth maintenance system
- Efficient solution: Justification Truth Maintenance Systems (JTMS)
  - We annotate every sentence in the knowledge base with a justification = set of sentences from which it was inferred
  - If we have $P \Rightarrow Q$ and we assert P then we can add Q with the justification $\{P, P \Rightarrow Q\}$
  - A sentence can have any number of justifications
  - If we retract P the JTMS will also retract the sentences for which P is a member of every justification.

$$\begin{array}{lcl} \{P, P \Rightarrow Q\} & \longrightarrow & Q \text{ retracted} \\ \{P, P \vee R \Rightarrow Q\} & \longrightarrow & Q \text{ retracted} \\ \{R, P \vee R \Rightarrow Q\} & \longrightarrow & Q \text{ not retracted} \end{array}$$

- A JTMS will actually mark sentences as “out” instead of retracting them
  - A sentence that is retracted might become pertinent again in the future
  - A JTMS will thus thus retain the whole inference chain should a justification become valid again
  - Bonus: JTMS also provide a mechanism for generating explanations

- Action A_t = leave for airport t minutes before flight
  - Will A_t get me there on time?
  - Problems:
    1. Partial observability (road state, other drivers' plans, . . .)
    2. Noisy sensors (traffic reports over the radio)
    3. Uncertainty in action outcomes (flat tire, . . .)
    4. Intractable complexity of modelling and predicting traffic
- A logical approach:
  - Risks falsehood: "A_{120} will get me there on time"
  - Leads to conclusions that are too weak for decision making: "A_{120} will get me there on time if there's no jam on Pont Champlain and it doesn't rain and my tires remain intact and . . . "
  - Note: I might reasonably expect that A_{1440} will get me there on time, but such a logical approach will make me spend a night in the airport

- Nonmonotonic/default logic: I assume won't get a flat tire, that there is no traffic jam on Champlain, etc

$$\mathit{drive}(\mathit{sherbrooke}, \mathit{dorval}, 120) \wedge \mathbb{M}\neg \mathit{flat\_tire} \Rightarrow A_{120}$$

$$\frac{\mathit{drive}(\mathit{sherbrooke}, \mathit{dorval}, 120) \quad : \quad \neg \mathit{jammed}(\mathit{champlain})}{A_{120}}$$

i.e., assume that A_{120} works unless contradicted by evidence

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But what assumptions are reasonable?

- Rules with fudge factors:

$$sprinkler \Rightarrow_{0.99} wet\_grass$$
$$wet\_grass \Rightarrow_{0.7} rained$$

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- Probability: given the available evidence, A_{120} will get me to the airport in time with probability 0.03

- Probability summarizes
  - Laziness to enumerate all the exceptions, facts, ...
  - Ignorance, i.e., lack of relevant facts, initial conditions, ...
- Bayesian (or subjective) probability relates probability to one's own state of knowledge

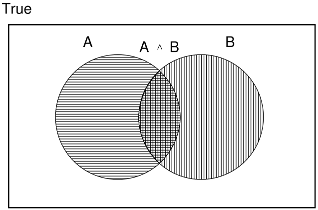
$$P(A_{120}|intact\_tires) = 0.06$$

  - Probabilities change with new evidence

$$P(A_{120}|intact\_tires \land 3am) = 0.75$$

  - Analogous to logical entailment ($KB \models \alpha$), not truth
  - Axioms of probability:

    1. $0 \leq P(A) \leq 1$
    2. $P(True) = 1$; $P(False) = 0$
    3. $P(A \lor B) = P(A) + P(B) - P(A \land B)$



Syntax

- Possible worlds defined by assignment of values to random variables
- Propositional (Boolean) random varianles: *Cavity* (do I have a cavity?)
  - including propositional logic expressions: $\neg Burglary \vee Earthquake$
- Multivalued random variables: *Weather* is one of $\langle sunny, rain, cloudy, snow \rangle$
  - Values must be exhaustive and mutually exclusive
- Propositions constructed by assignment of a value: $Weather = sunny$
- Unconditional (prior) probabilities of propositions: $P(Weather = sunny) = 0.72$
- Conditional (posterior) probabilities: $P(Cavity | Toothache) = 0.8$ (i.e., probability given that *Toothache* is all I know)

- Probability distribution gives values for all possible assignments: $\mathbb{P}(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$ (normalized)
- Joint probability distribution for a set of variables: gives values for each possible assignment to all the variables $\mathbb{P}(Weather, Cavity)$ = a 4×2 matrix of values:

$Weather =$	*sunny*	*rain*	*cloudy*	*snow*
$Cavity = true$				
$Cavity = false$				

- If we know more, e.g., *Cavity* is also given, then we have $P(Cavity|Toothache, Cavity) = 1$
- New evidence may be irrelevant, allowing simplification: $P(Cavity|Toothache, Midterm) = P(Cavity|Toothache) = 0.8$
- Conditional probability:

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \text{ if } P(B) \neq 0$$

  alternatively

$$P(A \wedge B) = P(A|B)P(B) = P(B|A)P(A)$$

- Bayes' rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

  Why is this useful?

$$
\begin{aligned}
P(\mathit{Meningitis} \mid \mathit{StiffNeck}) &= \frac{P(\mathit{StiffNeck} \mid \mathit{Meningitis})P(\mathit{Meningitis})}{P(\mathit{StiffNeck})} \\
&= \frac{0.8 \times 0.0001}{0.1} = 0.0008
\end{aligned}
$$

$$P(Meningitis|StiffNeck) = \frac{P(StiffNeck|Meningitis)P(Meningitis)}{P(StiffNeck)}$$
$$= \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

- Bayes' rule is useful for assessing diagnostic probability from causal probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

- Chain rule: successive application of the product rule (on joint probability distributions)

$$\begin{aligned}
\mathbb{P}(X_1, \ldots, X_n) &= \mathbb{P}(X_1, \ldots, X_{n-1})\mathbb{P}(X_n|X_1, \ldots, X_{n-1}) \\
&= \mathbb{P}(X_1, \ldots, X_{n-2})\mathbb{P}(X_{n_1}|X_1, \ldots, X_{n-2})\mathbb{P}(X_n|X_1, \ldots, X_{n-1}) \\
&= \ldots \\
&= \prod_{i=1}^{n} \mathbb{P}(X_i|X_1, \ldots, X_{i-1})
\end{aligned}$$

NORMALIZATION

- We want to compute a posterior distribution over A given $B = b$, and suppose A has possible values $\langle a_1, \ldots, a_m \rangle$.

$$\begin{aligned} P(A = a_1 | B = b) &= P(B = b | A = a_1) P(A = a_1) / P(B = b) \\ &\cdots \\ P(A = a_m | B = b) &= P(B = b | A = a_m) P(A = a_m) / P(B = b) \end{aligned}$$

$$\begin{aligned} \sum_i P(A = a_i | B = b) &= \left(\sum_i P(B = b | A = a_i) P(A = a_i) \right) / P(B = b) \\ 1 &= \left(\sum_i P(B = b | A = a_i) P(A = a_i) \right) / P(B = b) \\ 1/P(B = b) &= 1 / \sum_i P(B = b | A = a_i) P(A = a_i) \end{aligned}$$

$\rightarrow$ normalization factor α

- $\mathbb{P}(A|B = b) = \alpha \mathbb{P}(B = b|A)\mathbb{P}(A)$
  e.g., let $\mathbb{P}(B = b|A)\mathbb{P}(A) = \langle 0.4, 0.2, 0.2 \rangle$;
  then $\mathbb{P}(A|B = b) = \alpha \langle 0.4, 0.2, 0.2 \rangle = \frac{\langle 0.4, 0.2, 0.2 \rangle}{0.4 + 0.2 + 0.2} = \langle 0.5, 0.25, 0.25 \rangle$

- Often easier to analyze each specific circumstance instead of the whole situation:

$$\begin{aligned} & P(RunOver|Cross) \\ & \quad = \quad P(RunOver|Cross, Light = green)P(Light = green|Cross) \\ & \quad + \quad P(RunOver|Cross, Light = yellow)P(Light = yellow|Cross) \\ & \quad + \quad P(RunOver|Cross, Light = red)P(Light = red|Cross) \end{aligned}$$

- I.e., we can introduce a variable as an extra condition:

$$P(X|Y) = \sum_z P(X|Y, Z = z)P(Z = z|Y)$$

- When Y is absent, we have summing out or marginalization:

$$P(X) = \sum_z P(X|Z = z)P(Z = z) = \sum_z P(X, Z = z)$$

- Given a joint distribution over a set of variables, the distribution over any subset can be calculated by summing out the other variables

Full joint distribution

- A complete probability model specifies every entry in the joint distribution for all the variables $\mathbf{X} = X_1, \ldots, X_n$;
  - I.e., a probability for each possible world w_i.
  - Possible worlds are exclusive and exhaustive, hence the sum of the probabilities in the matrix is always 1: $\sum_i P(w_i) = 1$.

	$Toothache = true$	$Toothache = false$
$Cavity = true$	0.04	0.06
$Cavity = false$	0.01	0.89

- For any proposition ϕ defined on the random variables: $\phi(w_i)$ is true or false
  ϕ is equivalent to the disjunction of w_is where $\phi(w_i)$ is true, hence

$$P(\phi) = \sum_{w_i:\ \phi(w_i)} P(w_i)$$

  I.e., the unconditional probability of any proposition is computable as the sum of entries from the full joint distribution

- We are interested in the posterior joint distribution of the query variables $\mathbf{Y}$ given specific values $\mathbf{e}$ for the evidence variables $\mathbf{E}$.
- We may have hidden variables $\mathbf{H} = \mathbf{X} \setminus \mathbf{Y} \setminus \mathbf{E}$.
- Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbb{P}(\mathbf{Y}|\mathbf{E} = \mathbf{e}) = \alpha\mathbb{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbb{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$$

- The terms in the summation are joint entries because **Y**, **E**, and **H** together exhaust the set of random variables.
- Problem: Huge time and space complexity