CS 316: Utility

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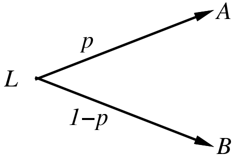
- An agent chooses among prizes (A, B, etc.) and lotteries, i.e., situations with uncertain prizes

Lottery $L = [p, A;\ (1-p), B]$



- Notation:

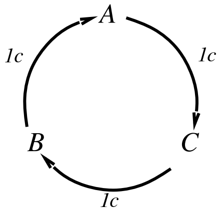
$A \succ B$	A preferred to B
$A \sim B$	indifference between A and B
$A \succsim B$	B not preferred to A

Rational preferences

- Preferences of a rational agent must obey constraints
  - Rational preferences $\Rightarrow$ behavior describable as maximization of expected utility
- Constraints:
  - Orderability: $(A \succ B) \vee (B \succ A) \vee (A \sim B)$
  - Transitivity: $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$
  - Continuity: $A \succ B \succ C \Rightarrow \exists p \ [p, A;\ 1 - p, C] \sim B$
  - Substitutability: $A \sim B \Rightarrow [p, A;\ 1 - p, C] \sim [p, B; 1 - p, C]$
  - Monotonicity: $A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A;\ 1 - p, B] \succsim [q, A;\ 1 - q, B])$

Rational preferences (cont'd)

- Violating the constraints leads to self-evident irrationality
- For example an agent with intransitive preferences can be induced to give away all its money
  - If $B \succ C$, then an agent who has C would pay (say) 1 cent to get B
  - If $A \succ B$, then an agent who has B would pay 1 cent to get A
  - If $C \succ A$, then an agent who has A would pay 1 cent to get C



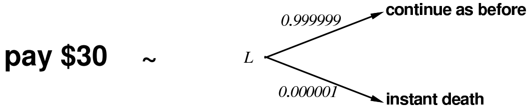
- Theorem (Ramsey, 1931; von Neumann and Morgenstern, 1944): Given preferences satisfying the constraints there exists a real-valued function U such that

$$U(A) \geq U(B) \Leftrightarrow A \succsim B$$

$$U([p_1, S_1; \ \ldots \ ; p_n, S_n]) = \sum_i p_i U(S_i)$$

- MEU principle: Choose the action that maximizes expected utility
- Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tictactoe

- Utilities map states to real numbers. Which numbers?
- Standard approach to assessment of human utilities: Compare a given state A to a standard lottery L_p that has
  - "best possible prize" $u_\top$ with probability p
  - "worst possible catastrophe" $u_\bot$ with probability $(1 - p)$
- adjust lottery probability p until $A \sim L_p$

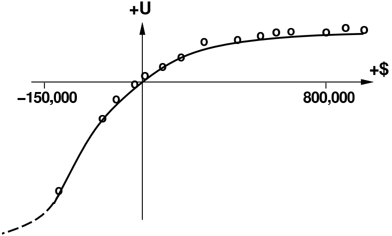


- Normalized utilities: $u_\top = 1.0$, $u_\perp = 0.0$
- Micromorts: one-millionth chance of death
  - useful for Russian roulette, paying to reduce product risks, etc.
- QALYs: quality-adjusted life years
  - useful for medical decisions involving substantial risk
- Note: behavior is invariant with respect to positive linear transformation

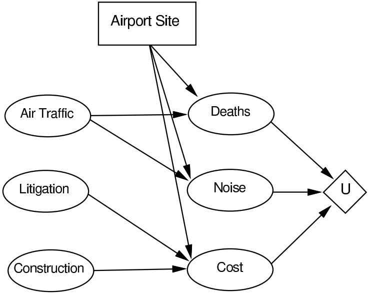
$$U'(x) = k_1 U(x) + k_2 \quad \text{where } k_1 > 0$$

- With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes

- Money does not behave as a utility function
- Given a lottery L with expected monetary value $EMV(L)$, usually $U(L) < U(EMV(L))$, i.e., people are risk-averse
- Utility curve: for what probability p am I indifferent between a fixed prize x and a lottery $[p, \$M;\ (1-p), \$0]$ for large M?
- Typical empirical data, extrapolated with risk-prone behavior:



- Add action nodes and utility nodes to belief networks to enable rational decision making

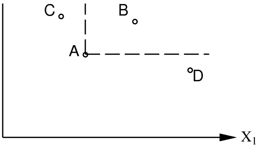


- Algorithm:
  For each value of action node:
  compute expected value of utility node given action, evidence
  Return MEU action

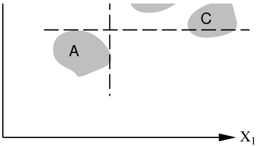
Multiattribute utility

- How can we handle utility functions of many variables $X_1 \ldots X_n$? E.g., what is $U(Deaths, Noise, Cost)$?
- How can complex utility functions be assessed from preference behaviour?
  - Idea 1: identify conditions under which decisions can be made without complete identification of $U(x_1, \ldots, x_n)$
  - Idea 2: identify various types of independence in preferences and derive consequent canonical forms for $U(x_1, \ldots, x_n)$

- Typically define attributes such that U is monotonic in each
- Strict dominance: choice B strictly dominates choice A iff
  $\forall\, i \;\; X_i(B) \geq X_i(A)$ (and hence $U(B) \geq U(A)$)



Deterministic attributes



Uncertain attributes

- Strict dominance seldom holds in practice

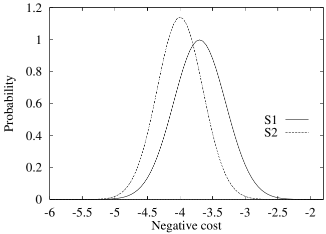
- Distribution p_1 stochastically dominates distribution p_2 iff

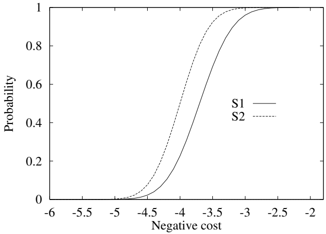
$$\forall t \quad \int_{-\infty}^{t} p_1(x)dx \leq \int_{-\infty}^{t} p_2(x)dx$$

- If U is monotonic in x, then A_1 with outcome distribution p_1 stochastically dominates A_2 with outcome distribution p_2:

$$\int_{-\infty}^{\infty} p_1(x)U(x)dx \geq \int_{-\infty}^{\infty} p_2(x)U(x)dx$$

- Multiattribute case: stochastic dominance on all attributes $\Rightarrow$ optimal





Stochastic dominance (cont'd)

- Stochastic dominance can often be determined without exact distributions using qualitative reasoning
  - E.g., construction cost increases with distance from city
    S_2 is further from the city than S_1
    $\Rightarrow S_1$ stochastically dominates S_2 on cost
  - E.g., injury increases with collision speed
- Can annotate belief networks with stochastic dominance information: $X \xrightarrow{+} Y$ (X positively influences Y) means that for every value $\mathbf{z}$ of Y's other parents $\mathbf{Z}$:

$$\forall x_1, x_2 \;\; x_1 \geq x_2 \Rightarrow \mathbb{P}(Y|x_1, \mathbf{z}) \text{ stochastically dominates } \mathbb{P}(Y|x_2, \mathbf{z})$$

- X_1 and X_2 preferentially independent of X_3 iff preference between $\langle x_1, x_2, x_3\rangle$ and $\langle x_1', x_2', x_3\rangle$ does not depend on x_3
- E.g., $\langle Noise, Cost, Safety\rangle$:
  ⟨20,000 suffer, $4.6 billion, 0.06 deaths/mpm⟩ vs.
  ⟨70,000 suffer, $4.2 billion, 0.06 deaths/mpm⟩
- Theorem (Leontief, 1947): if every pair of attributes is P.I. of its complement, then every subset of attributes is P.I of its complement: mutual P.I..
- Theorem (Debreu, 1960): mutual P.I. $\Rightarrow \exists$ additive value function:

$$V(S) = \sum_i V_i(X_i(S))$$

$$V(noise, cost, death) = -noise \times 10^4 - cost - deaths \times 10^{12}$$

- Hence assess n single-attribute functions; often a good approximation

- Need to consider preferences over lotteries:
  X is utility-independent of **Y** iff
  preferences over lotteries **X** do not depend on **Y**
- Mutual U.I.: each subset is U.I of its complement
  $\Rightarrow \exists$ multiplicative utility function:
  $U = k_1 U_1 + k_2 U_2 + k_3 U_3$
  $+ k_1 k_2 U_1 U_2 + k_2 k_3 U_2 U_3 + k_3 k_1 U_3 U_1$
  $+ k_1 k_2 k_3 U_1 U_2 U_3$
- Routine procedures and software packages for generating preference tests to identify various canonical families of utility functions

Value of information: Simple example

- One of the most important part of decision making: know what questions to ask
- Idea: compute value of acquiring each possible piece of evidence Can be done directly from decision network
- Example: buying oil drilling rights
  - Two blocks A and B, exactly one has oil, worth k
  - Prior probabilities 0.5 each, mutually exclusive
  - Current price of each block is $k/2$
  - Consultant offers accurate survey of A. Fair price?
- Solution: compute expected value of information
  = expected value of best action given the information
  minus expected value of best action without information
  Survey may say "oil in A" or "no oil in A", prob. 0.5 each
  = [0.5 $\times$ value of "buy A" given "oil in A"
  + 0.5 $\times$ value of "buy B" given "no oil in A"] – 0
  = $(0.5 \times k/2) + (0.5 \times k/2) - 0 = k/2$

- Current evidence E, current best action α
- Possible action outcomes S_i, potential new evidence E_j

$$EU(\alpha|E) = \max_a \sum_i U(S_i)\ P(S_i|E, a)$$

- Suppose we knew $E_j = e_{jk}$, then we would choose $\alpha_{e_{jk}}$ s.t.

$$EU(\alpha_{e_{jk}}|E, E_j = e_{jk}) = \max_a \sum_i U(S_i)\ P(S_i|E, a, E_j = e_{jk})$$

- E_j is a random variable whose value is *currently* unknown
  $\Rightarrow$ must compute expected gain over all possible values:

$$VPI_E(E_j) = \left(\sum_k P(E_j = e_{jk}|E) EU(\alpha_{e_{jk}}|E, E_j = e_{jk})\right) - EU(\alpha|E)$$

(VPI = value of perfect information)

Properties of VPI

- Nonnegative—in *expectation*, not *post hoc*

$$\forall j, E \;\; VPI_E(E_j) \geq 0$$

- Nonadditive—consider, e.g., obtaining E_j twice

$$VPI_E(E_j, E_k) \neq VPI_E(E_j) + VPI_E(E_k)$$

- Order-independent

$$VPI_E(E_j, E_k) = VPI_E(E_j) + VPI_{E,E_j}(E_k) = VPI_E(E_k) + VPI_{E,E_k}(E_j)$$

- Note: when more than one piece of evidence can be gathered, maximizing VPI for each to select one is not always optimal

  $\Rightarrow$ evidence-gathering becomes a sequential decision problem

QUALITATIVE BEHAVIORS

a) Choice is obvious, information worth little
b) Choice is nonobvious, information worth a lot
c) Choice is nonobvious, information worth little