# Asymptotic Order Notations

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# TIME ANALYSIS

- Decide on the basic operation(s) then count
- Simple examples:
  1. **for** $i = 1$ **to** $n$ **do**
     $sum \leftarrow sum + A_i$

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     - With basic operation addition: $t(n) \approx n$
  2. // bubble sort
     **for** $i = 1$ **to** $n$ **do**
        **for** $j = i + 1$ **to** $n$ **do**
           **if** $A_j < A_i$ **then** $A_j \leftrightarrow A_i$

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       - **if** $A_j < A_i$ **then** $A_j \leftrightarrow A_i$
     - With basic operation swap: $t(n) \approx n^2$
  3. // matrix multiplication ($C = A \times B$)
     **for** $i = 1$ **to** $n$ **do**
     - **for** $j = i$ **to** $n$ **do**
       - $C_{i,j} \leftarrow 0$ **for** $k = 1$ **to** $n$ **do**
         - $C_{i,j} \leftarrow C_{i,j} + A_{i,k} \times B_{k,j}$

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           $C_{i,j} \leftarrow C_{i,j} + A_{i,k} \times B_{k,j}$
     - With multiplication as basic operation: $t(n) \approx n^3$

- **algorithm** SEQSEARCH($x$, $A$, $n$):

```
i ← 1
found ← false
while ¬found ∧ i ≤ n do
    if A_i = x then
        found ← true
        return i
    else i ← i + 1
return −1
```

  - Best case ($x$ first):
  - Worst case ($x$ last or not in the sequence):

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  - Best case ($x$ first): $t(n) \in O(1)$
  - Worst case ($x$ last or not in the sequence): $t(n) \in O(n)$
  - Average case:
    - Assuming that $x$ is in the list and equally likely to be in any position: $t(n) \approx \sum_{k=1}^{n} k \times 1/n = 1/n \sum_{k=1}^{n} k = (1/n)(n(n+1)/2) = (n+1)/2 \in O(n)$

- **algorithm** SeqSearch($x$, $A$, $n$):

```
i ← 1
found ← false
while ¬found ∧ i ≤ n do
    if A_i = x then
        found ← true
        return i
    else i ← i + 1
return −1
```

  - Best case ($x$ first): $t(n) \in O(1)$
  - Worst case ($x$ last or not in the sequence): $t(n) \in O(n)$
  - Average case:
    - Assuming that $x$ is in the list and equally likely to be in any position: $t(n) \approx \sum_{k=1}^{n} k \times 1/n = 1/n \sum_{k=1}^{n} k = (1/n)(n(n+1)/2) = (n+1)/2 \in O(n)$
    - Assuming $x$ is in the list with probability $p$: $t(n) \approx p \times \sum_{k=1}^{n} k \times 1/n + (1-p) \times n = p \times (n+1)/2 + (1-p) \times n = pn/2 + p/2 + n - pn = n(1-p/2) + p/2 \in O(n)$

# ANYMPTOTIC TIME ANALYSIS

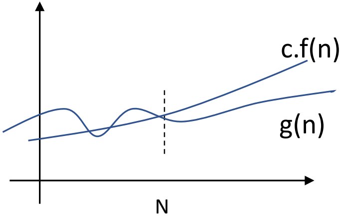
- Behaviour of the algorithm (running time) for arbitrarily large input size
- Suppose:

$$
\begin{aligned}
t_1(n) &= 10n \\
t_2(n) &= 0.1n^2 \\
t_3(n) &= 0.1n^2 + n + 100
\end{aligned}
$$

| $n$ | $10n$ | | $0.1n^2$ | | $0.1n^2 + n + 100$ |
|---|---|---|---|---|---|
| 1 | 10 | | 0.1 | | 101.1 |
| 10 | 100 | $>$ | 10 | $<$ | 120 |
| 20 | 200 | | 40 | | 160 |
| 50 | 500 | | 250 | | 400 |
| 100 | 1000 | | 1000 | | 1200 |
| 1000 | 10,000 | $<$ | 100,000 | $\approx$ | 110,100 |
| 10,000 | 100,000 | | 10,000,000 | | 10,100,100 |

- Asymptotic upper bound
- Several (equivalent) definitions:
  - $O(f(n))$ is the set of exactly all the functions that grow at most as fast as $f(n)$
    - The set of all functions with a smaller or equal rate of growth than $f(n)$
  - $g(n) \in O(f(n))$ iff $g(n)$ is bounded above by $f(n)$ except for a constant factor and a finite number of exceptions
  - Formally:

$$O(f(n)) = \{ \quad g(n) : \exists\, c > 0, N \geq 0 : \forall\, n \geq N : g(n) \leq c \times f(n) \quad \}$$



# BIG-OH EXAMPLES

- $5n^2 \in O(n^2)$?
  - $\iff \forall\, n \geq N : 5n^2 \leq c \times n^2$

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  - $\iff \forall n \geq N : 5n^2 \leq c \times n^2$
  - True for $c = 5$ and $N = 0$
- $n \in O(n^2)$
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- $n \in O(n^2)$
  - $\iff \forall n \geq N : n \leq c \times n^2$
  - True for $c = 1$ and $N = 1$
- $n^2 \in O(n^2 + 10n)$
  - $\iff \forall n \geq N : n^2 \leq c \times (n^2 + 10n)$

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  - $\iff \forall n \geq N : n \leq c \times n^2$
  - True for $c = 1$ and $N = 1$
- $n^2 \in O(n^2 + 10n)$
  - $\iff \forall n \geq N : n^2 \leq c \times (n^2 + 10n)$
  - Take $c = 1$
  - $\forall n \geq N : n^2 \leq n^2 + 10n$?
  - But $\forall n \geq N : 0 \leq 10n$
  - So true for $N = 0$
- $n^2 + 10n \in O(n^2)$
  - $\iff \forall n \geq N : n^2 + 10n \leq c \times n^2$
  - $\iff \forall n \geq N : n + 10 \leq c \times n$

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- $n \in O(n^2)$
  - $\iff \forall n \geq N : n \leq c \times n^2$
  - True for $c = 1$ and $N = 1$
- $n^2 \in O(n^2 + 10n)$
  - $\iff \forall n \geq N : n^2 \leq c \times (n^2 + 10n)$
  - Take $c = 1$
  - $\forall n \geq N : n^2 \leq n^2 + 10n$?
  - But $\forall n \geq N : 0 \leq 10n$
  - So true for $N = 0$
- $n^2 + 10n \in O(n^2)$
  - $\iff \forall n \geq N : n^2 + 10n \leq c \times n^2$
  - $\iff \forall n \geq N : n + 10 \leq c \times n$
  - But $n + 10 \leq n + 10n = 11n$
  - So true for $c = 11$ and $N = 1$

# Big-Oh examples (cont'd)

- $n(n-1)/2 \in O(n^2)$ ?
  - $\iff \forall n \geq N : n(n-1)/2 \leq c \times n^2$
  - $\iff \forall n \geq N : (n-1)/2 \leq c \times n$
  - $\iff \forall n \geq N : n - 1 \leq c \times 2n$

# BIG-OH EXAMPLES (CONT'D)

- $n(n-1)/2 \in O(n^2)$ ?
  - $\iff \forall n \geq N : n(n-1)/2 \leq c \times n^2$
  - $\iff \forall n \geq N : (n-1)/2 \leq c \times n$
  - $\iff \forall n \geq N : n-1 \leq c \times 2n$
  - Take $c = 0.5$:
  - $\iff \forall n \geq N : n-1 \leq n \iff 1 \geq 0$
  - True for $c = 0.5$ and $N = 0$
- $2n + 3\log n \in O(n)$
  - $\iff \forall n \geq N : 2n + 3\log n \leq c \times n$

# Big-Oh examples (cont'd)

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  - Take $c = 0.5$:
  - $\iff \forall n \geq N : n-1 \leq n \iff 1 \geq 0$
  - True for $c = 0.5$ and $N = 0$
- $2n + 3 \log n \in O(n)$
  - $\iff \forall n \geq N : 2n + 3 \log n \leq c \times n$
  - We know $\log n \leq n$ and so $2n + 3 \log n \leq 2n + 3n = 5n$
  - So true for $c = 5$ and $N = 1$
- $n^2/2 \in O(n)$?
  - $\iff \forall n \geq N : n^2/2 \leq c \times n$
  - $\iff \forall n \geq N : n \leq 2c$

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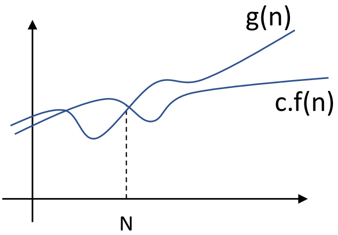
  - So true for $c = 5$ and $N = 1$
- $n^2/2 \in O(n)$?
  - $\iff \forall n \geq N : n^2/2 \leq c \times n$
  - $\iff \forall n \geq N : n \leq 2c$
  - Cannot be true for unbounded $n$ so false
- Can pick the constant $c$ but cannot pick any particular $n$
  - In particular, the conclusion must be true for an unbounded $n$

# THE BIG-$\Omega$ NOTATION

- Asymptotic lower bound
- Several (equivalent) definitions:
  - $\Omega(f(n))$ is a set that includes exactly all the functions that grow at least as fast as $f(n)$
    - The set of all functions with a larger or equal rate of growth than $f(n)$
  - $g(n) \in \Omega(f(n))$ iff $g(n)$ is bounded below by $f(n)$ except for a constant factor and a finite number of exceptions
  - Formally:

$$\Omega(f(n)) = \{ \quad g(n) : \exists\, c > 0, N \geq 0 : \forall\, n \geq N : g(n) \geq c \times f(n) \quad \}$$



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  - $\iff \forall\, n \geq N : n^2 \geq c \times n$
  - True for $c = 1$ and $N = 1$
- $5n–3 \in \Omega(n)$?
  - $\iff \forall\, n \geq N : 5n - 3 \geq c \times n$

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  - $\iff \forall n \geq N : n^2 \geq c \times n$
  - True for $c = 1$ and $N = 1$
- $5n–3 \in \Omega(n)$?
  - $\iff \forall n \geq N : 5n - 3 \geq c \times n$
  - Take $c = 1$
  - $\iff \forall n \geq N : 4n \geq 3$
  - True for $N = 1$
- $n(n-1)/2 \in \Omega(n^2)$?
  - $\iff \forall n \geq N : n(n-1)/2 \geq c \times n^2$
  - $\iff \forall n \geq N : n - 1 \geq 2c \times n$

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  - $\iff \forall n \geq N : n(n-1)/2 \geq c \times n^2$
  - $\iff \forall n \geq N : n - 1 \geq 2c \times n$
  - We know that $\forall n \geq 2 : n - 1 \geq n/2$ ($n - 1 \geq n/2 \Rightarrow 2n - 2 \geq n \Rightarrow n \geq 2$)
  - So true for $2c = 1/2$ that is, $c = 1/4$ and $N = 2$
- $2n \in \Omega(n^2/2)$?
  - $\iff \forall n \geq N : 2n \geq c \times n^2/2$
  - $\iff \forall n \geq N : 4 \geq cn$

# Big-Ω examples

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- $n^2 \in \Omega(n)$?
  - $\iff \forall n \geq N : n^2 \geq c \times n$
  - True for $c = 1$ and $N = 1$
- $5n–3 \in \Omega(n)$?
  - $\iff \forall n \geq N : 5n - 3 \geq c \times n$
  - Take $c = 1$
  - $\iff \forall n \geq N : 4n \geq 3$
  - True for $N = 1$
- $n(n-1)/2 \in \Omega(n^2)$?
  - $\iff \forall n \geq N : n(n-1)/2 \geq c \times n^2$
  - $\iff \forall n \geq N : n - 1 \geq 2c \times n$
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  - So true for $2c = 1/2$ that is, $c = 1/4$ and $N = 2$
- $2n \in \Omega(n^2/2)$?
  - $\iff \forall n \geq N : 2n \geq c \times n^2/2$
  - $\iff \forall n \geq N : 4 \geq cn$
  - Cannot be true for unbounded $n$ so false

- Asymptotic tight bound
- Several (equivalent) definitions:
  - $\Theta(f(n))$ is a set that includes exactly all the functions that grow as fast as $f(n)$
    - The set of all functions with an equal rate of growth to $f(n)$
  - $g(n) \in \Theta(f(n))$ iff $g(n)$ is bounded above and below by $f(n)$ except for a constant factor and a finite number of exceptions
  - Formally:

$$\Omega(f(n)) = \{ \quad g(n) : \exists\, c_1 > 0, c_2 > 0, N \geq 0 : \forall\, n \geq N : c_1 \times f(n) \leq g(n) \leq c_2 \times f(n) \quad \}$$

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- Several (equivalent) definitions:
  - $\Theta(f(n))$ is a set that includes exactly all the functions that grow as fast as $f(n)$
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  - Formally:

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- Therefore $\Theta(f(n)) = O(f(n)) \cap \Omega(f(n))$
  - In order to prove that $g(n) \in \Theta(f(n))$ one can prove that $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$
  - Example: $n(n-1)/2 \in \Theta(n^2)$ since we already showed earlier that $n(n-1)/2 \in O(n^2)$ and $n(n-1)/2 \in \Omega(n^2)$

# TWO USEFUL PROPERTIES OF $\Theta$

## Theorem (complexity of polynomials)

*Let $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n^1 + a_0$ be any polynomial in $n$ of degree $k$ with $a_k > 0$. Then $p(n) \in \Theta(n^k)$.*

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- We show that $p(n) \in O(n^k)$ by choosing $c = (k+1) \times \max_{i=1}^{k} a_i$ and noting that $(\max_{i=1}^{k} a_i) n^k \geq a_j n^k \geq a_j n^j$ for all $0 \leq j \leq k$

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- To show that $p(n) \in \Omega(n^k)$ we note that $p(n) = \frac{a_k}{2} n^k + \sum_{j=0}^{k-1} \left(\frac{a_k}{2k} n^k + a_j n^j\right)$, we also note that $\left(\frac{a_k}{2k} n^k + a_j n^j\right) \geq 0$ for large enough $n$ ($n \geq \max_{j=0}^{k-1} -2ka_j/a_k$), so $p(n) \geq \frac{a_k}{2} n^k$, and so $c = a_k/2k$

## Theorem (complexity of polynomials)

*Let $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n^1 + a_0$ be any polynomial in $n$ of degree $k$ with $a_k > 0$. Then $p(n) \in \Theta(n^k)$.*

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- Examples:
  - $n(n-1)/2 \in \Theta(n^2)$ (also shown without the theorem earlier)
  - $2n^2 - 3n \in \Theta(n^2)$

# TWO USEFUL PROPERTIES OF $\Theta$

## Theorem (complexity of polynomials)

*Let* $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n^1 + a_0$ *be any polynomial in* $n$ *of degree* $k$ *with* $a_k > 0$. *Then* $p(n) \in \Theta(n^k)$.

- We show that $p(n) \in O(n^k)$ by choosing $c = (k+1) \times \max_{i=1}^{k} a_i$ and noting that $(\max_{i=1}^{k} a_i) n^k \geq a_j n^k \geq a_j n^j$ for all $0 \leq j \leq k$
- To show that $p(n) \in \Omega(n^k)$ we note that $p(n) = \frac{a_k}{2} n^k + \sum_{j=0}^{k-1} \left(\frac{a_k}{2k} n^k + a_j n^j\right)$, we also note that $\left(\frac{a_k}{2k} n^k + a_j n^j\right) \geq 0$ for large enough $n$ ($n \geq \max_{j=0}^{k-1} -2ka_j/a_k$), so $p(n) \geq \frac{a_k}{2} n^k$, and so $c = a_k/2k$
- Examples:
  - $n(n-1)/2 \in \Theta(n^2)$ (also shown without the theorem earlier)
  - $2n^2 - 3n \in \Theta(n^2)$

## Theorem (logarithm base change)

$\log_a n = \Theta(\log_b n) = \Theta(\log n)$ *for any* $a, b > 1$

- $o(f(n))$ is the set of exactly all the functions with a strictly smaller rate of growth than $f(n)$
  - Formally:

$$o(f(n)) = \{ \quad g(n) : \forall c > 0 : \exists N \geq 0 : \forall n \geq N : g(n) \leq c \times f(n) \quad \}$$

- $\omega(f(n))$ is the set of exactly all the functions with a strictly larger rate of growth than $f(n)$
  - Formally:

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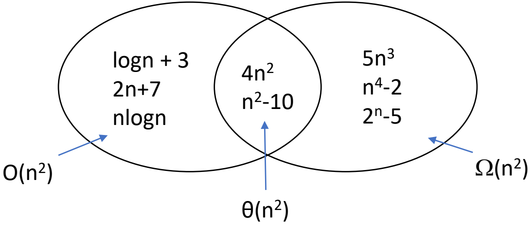
# Rules and properties

| Order | Rate of growth of the two functions |
|---|---|
| $O$ | $\leq$ |
| $\Omega$ | $\geq$ |
| $\Theta$ | $=$ |
| $o$ | $<$ |
| $\omega$ | $>$ |

Venn diagram for orders:




$$g(n) \in \Theta(f(n)) \quad \text{iff} \quad f(n) \in \Theta(g(n))$$
$$g(n) \in O(f(n)) \quad \text{iff} \quad f(n) \in \Omega(g(n))$$
$$g(n) \in o(f(n)) \quad \text{iff} \quad f(n) \in \omega(g(n))$$
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$$g(n) \in \Theta(f(n)) \quad \text{iff} \quad g(n) \in O(f(n)) \cap \Omega(f(n))$$
$$\text{If} \quad g(n) \in o(f(n)) \quad \text{then} \quad g(n) \in O(f(n)) \setminus \Omega(f(n))$$
$$\text{If} \quad g(n) \in \omega(f(n)) \quad \text{then} \quad g(n) \in \Omega(f(n)) \setminus O(f(n))$$

All orders are transitive:

$$\forall \mathbb{X} \in \{O, \Omega, \Theta, o, \omega\} : f(n) \in \mathbb{X}(g(n)) \wedge g(n) \in \mathbb{X}(h(n)) \Rightarrow f(n) \in \mathbb{X}(h(n))$$

# RULES AND PROPERTIES (CONT'D)

- $O(f(n)) + O(g(n)) = O(f(n) + g(n))$
  - For example $O(n^2) + O(n) = O(n^2 + n)$ $(= O(n^2))$

# Rules and properties (cont'd)

- $O(f(n)) + O(g(n)) = O(f(n) + g(n))$
  - For example $O(n^2) + O(n) = O(n^2 + n)$ $(= O(n^2))$
  - Note: $O(n) + O(n) = O(n)$ (since $O(2n) = O(n)$)
  - Subtraction is trickier: $O(n) - O(n) = O(n)$!
- $O(f(n)) \times O(g(n)) = O(f(n) \times g(n))$
  - For example $O(n^2) \times O(n) = O(n^3)$
- $n \times O(f(n)) = O(n \times f(n))$
  - For example $\underbrace{O(n) + O(n) + ... + O(n)}_{n \text{ times}} = O(n^2)$
- $g(n) \in o(f(n))$ as long as $g(n)$ is in a set to the left of the set that includes $f(n)$ in the following list (with $k > j > 2$ and $b > a > 1$):

$$\Theta(\log n) \quad \Theta(n) \quad \Theta(n \log n) \quad \Theta(n^2) \quad \Theta(n^j) \quad \Theta(n^k) \quad \Theta(a^n) \quad \Theta(b^n) \quad \Theta(n!)$$

- Orders can be used in equations, with an implicit existential quantifier
  Examples:
  - $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means: there exists a function $f(n) \in \Theta(n)$ such that $2n^2 + 3n + 1 = 2n^2 + f(n)$
  - $2n^2 + \Theta(n) = \Theta(n^2)$ means: there exist functions $f(n) \in \Theta(n)$ and $g(n) \in \Theta(n^2)$ such that $2n^2 + f(n) = g(n^2)$

# USING LIMITS

- It is not always obvious how to prove a complexity claim
  - Especially when dealing with $o$ and $\omega$, which requires a proof for all constants
- We can then retort to limits:

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \begin{cases} c & \Rightarrow & g(n) \in \Theta(f(n)) \\ 0 & \Rightarrow & g(n) \in o(f(n)) \\ \infty & \Rightarrow & g(n) \in \omega(f(n)) \end{cases}$$

- When $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is not defined (e.g., $\infty/\infty$) we can apply the l'Hôpital rule:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

  - Note in passing that all the complexity functions satisfy the prerequisites of the l'Hôpital rule

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  - Therefore $n^2 \log n \in o((n \log n)^2) \Rightarrow n^2 \log n \in O((n \log n)^2)$
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- $n \log n \in O(n^2)$?
  - $\lim_{n\to\infty} \frac{n \log n}{n^2} = \lim_{n\to\infty} \frac{\log n}{n} = \frac{\infty}{\infty}$, inconclusive
  - Apply l'Hôpital: $\lim_{n\to\infty} \frac{\log_b n}{n} = \lim_{n\to\infty} \frac{1/(n \ln b)}{1} = \frac{1}{n \ln b} = 0$
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  - Therefore $n \log n \in o(n^2) \Rightarrow n \log n \in O(n^2)$
- $n^n \in ?(10^n)$
  - $\lim_{n\to\infty} \frac{n^n}{10^n} = \lim_{n\to\infty} (n/10)^n = \infty^\infty = \infty$
  - Therefore $n^n \in \omega(10^n)$ and also $n^n \in \Omega(10^n)$

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# EXAMPLES OF USING LIMITS (CONT'D)

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  - Useful properties: $\log_b a = \frac{\log_x a}{\log_x b}$ and $\log_a b = \frac{1}{\log_b a}$
  - We have $\log_2 n = \frac{\log_8 n}{\log_8 2} = \log_8 n \log_2 8 = 3 \log_8 n$, so $8^{\log_2 n} = n^3$
  - $\lim_{n\to\infty} \frac{n^3}{n^2} = \lim_{n\to\infty} n = \infty$ so $8^{\log_2 n} \in \omega(2^n)$ (and $8^{\log_2 n} \in \Omega(2^n)$)

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**Corollary (of the logarithm base change)**

*For $a, b > 1$ $a^{\log_b(n)} = O(n^k)$ with $k = \lceil 1/\log_b a \rceil$*