Backtracking

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CS 317, Fall 2024

When dynamic programming does not work

- We use backtracking
  - Commonly used to make a sequence of decisions to build a recursively defined solution satisfying given constraints
  - In each recursive call we make exactly one decision which is consistent with all the previous decisions

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  - Example:

algorithm RecKnapsack(i, C, n, p, w): (handle the i-th object)
  if $i > n$ **then** **return** $(0, \langle\rangle)$
  else
    $(p_-, X_-) \leftarrow$ RecKnapsack($i+1, C, n, p, w$) (do not pick item i)
    if $w_i \leq C$ **then**
      $(p_+, X_+) \leftarrow$ RecKnapsack($i+1, C-w_i, n, p, w$) (pick item i if we can)
    else
      $(p_+, X_+) \leftarrow (0, \langle\rangle)$
    return MaxFst($\{(p_-, \langle 0\rangle + X_-), (p_+ + w_i, \langle 1\rangle + X_+)\}$)

WHEN DYNAMIC PROGRAMMING DOES NOT WORK

- We use backtracking
  - Commonly used to make a sequence of decisions to build a recursively defined solution satisfying given constraints
  - In each recursive call we make exactly one decision which is consistent with all the previous decisions
  - Example:

```
algorithm RecKnapsack(i, C, n, p, w):        (handle the i-th object)
    if i > n then return (0, ⟨⟩)
    else
        (p−, X−) ← RecKnapsack(i + 1, C, n, p, w)     (do not pick item i)
        if w_i ≤ C then
            (p+, X+) ← RecKnapsack(i+1, C − w_i, n, p, w)     (pick item i if we can)
        else
            (p+, X+) ← (0, ⟨⟩)
        return MaxFst({(p−, ⟨0⟩ + X−), (p+ + w_i, ⟨1⟩ + X+)})
```

- Alternative to backtracking: brute force
  - Generate all possible complete sequences of decisions one by one and check if they yield a solution
  - Backtracking has a chance of doing better since it stops when a sequence is hopeless
  - Example: Generate all n-digits in lexicographic order, check that each such a number yields the optimal 0/1 Knapsack solution

n-QUEENS

- Given an $n \times n$ chess board, and n queens, place each i-th queen on the i-th row so that no two queens check each other
  - Intermediate result: $\langle x_1, x_2, \ldots, x_i \rangle$, $i \leq n$
  - Constraints: x_i and x_k, $j \neq k$ are neither the same nor on the same diagonal
  - Decision: placement of one more queen
- Brute force: generate and then check all the possible sequences $\langle x_1, x_2, \ldots, x_n \rangle \rightarrow \Theta(n^{n+1})$ time
- Backtracking:

```
algorithm QUEENS(⟨x1, x2, ..., xi⟩):
    if i = n then return ⟨x1, x2, ..., xn⟩
    else
        for j = 1 to n do
            if PROMISING(⟨x1, x2, ..., xi, j⟩)
            then
                QUEENS(⟨x1, x2, ..., xi, j⟩)
```

```
algorithm PROMISING(⟨x1, x2, ..., xi⟩):
    k ← 1
    safe ← TRUE
    while k < i ∧ safe do
        if xi = xk ∨ |xi − xk| = i − k then
            safe ← FALSE
        k ← k + 1
    return safe
```

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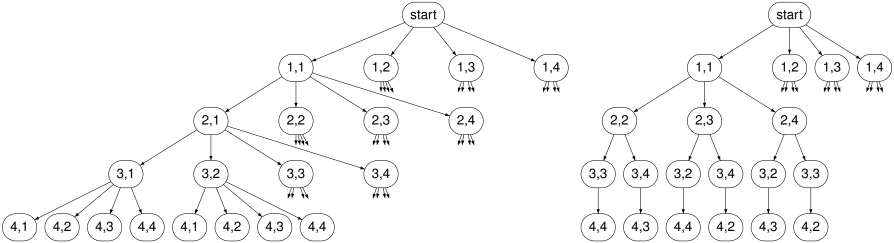
```

- Common patterns:
  - Traverse tree of states (aka state space)
    - Different decisions yield different next states
  - Carry over enough information between recursive calls to check feasibility

- Whole state space ($n = 4$): $4^4 = 256$ leaves and $1 + 4 + 4^2 + 4^3 + 4^4 = 341$ nodes
  - Slight optimization of the state space: no two queens can be on the same column ($1 + 4 + 4 \times 3 + 4 \times 3 \times 2 + 4 \times 3 \times 2 \times 1 = 65$ nodes)
  - Backtracking expands only 61 nodes



- For $n = 8$ we have $19,173,961$ nodes overall, $109,601$ optimized, and $15,721$ expanded by backtracking

$$A_{i,j} = \begin{cases} p_i \text{ (root } i) & \text{if } i = j \\ \min_{i \le k \le j}(A_{i,k-1} + A_{k+1,j} + \sum_{m=i}^{j} p_m) \text{ (root } k) & \text{if } i < j \end{cases}$$

- Brute force: generate all possible trees, retain the optimal one
- Backtracking for the optimal cost:

```
algorithm CostBST(i, j):
    if i = j then return p_i
    else if i > j then return 0
    else
        m ← ∞
        for k = i to j do
            b ← CostBST(i, k − 1)
            c ← CostBST(k + 1, j)
            a ← b + c + Σ_{m=i}^{j} p_m
            if a < m then
                m ← a
        return m
```

$$A_{i,j} = \begin{cases} p_i \text{ (root } i\text{)} & \text{if } i = j \\ \min_{i \le k \le j}(A_{i,k-1} + A_{k+1,j} + \sum_{m=i}^{j} p_m) \text{ (root } k\text{)} & \text{if } i < j \end{cases}$$

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      a ← b + c + Σ_{m=i}^{j} p_m
      if a < m then
        m ← a
    return m
```

- Backtracking for the optimal BST:

```
algorithm OptBST(i, j):
  if i = j then return (p_i, Node(i))
  else if i > j then return (0, Null)
  else
    m ← (∞, Null)
    for k = i to j do
      (b, l) ← OptBST(i, k − 1)
      (c, r) ← OptBST(k + 1, j)
      a ← b + c + Σ_{m=i}^{j} p_m
      if a < m then
        m ← (a, Node(k, l, r))
    return m
```

$$A_{i,j} = \begin{cases} p_i \text{ (root } i\text{)} & \text{if } i = j \\ \min_{i \le k \le j}(A_{i,k-1} + A_{k+1,j} + \sum_{m=i}^{j} p_m) \text{ (root } k\text{)} & \text{if } i < j \end{cases}$$

- Brute force: generate all possible trees, retain the optimal one
- Backtracking for the optimal cost:

```
algorithm COSTBST(i, j):
    if i = j then return p_i
    else if i > j then return 0
    else
        m ← ∞
        for k = i to j do
            b ← COSTBST(i, k − 1)
            c ← COSTBST(k + 1, j)
            a ← b + c + Σ_{m=i}^{j} p_m
            if a < m then
                m ← a
        return m
```

- Backtracking for the optimal BST:

```
algorithm OPTBST(i, j):
    if i = j then return (p_i, NODE(i))
    else if i > j then return (0, NULL)
    else
        m ← (∞, NULL)
        for k = i to j do
            (b, l) ← OPTBST(i, k − 1)
            (c, r) ← OPTBST(k + 1, j)
            a ← b + c + Σ_{m=i}^{j} p_m
            if a < m then
                m ← (a, NODE(k, l, r))
        return m
```

- When we solve a problem using backtracking we effectively solve a whole family of related problems

TRAVELING SALESMAN

- Brute force: try all the permutations, retain the one with minimal cost
- Backtracking: With $g(i, S)$ the length of the shortest path starting at i and going through all the vertices in S back to 1,

$$g(i,S) = \begin{cases} \min_{(i,j)\in E}(w(i,j)) & \text{if } S = \emptyset \\ \min_{j\in S}(w((i,j)) + g(j, S\setminus\{j\})) & \text{otherwise} \end{cases}$$

```
algorithm TS(i, S):
    if S = ∅ then
        return min_{(i,j)∈E}(w(i,j))
    else
        m ← ∞
        forall j ∈ S do
            a ← w((i,j)) + TS(j, S \ {j})
            if a < m then
                m ← a
        return m
```

```
algorithm TSX(i, S):
    if S = ∅ then
        return (min_{(i,j)∈E}(w(i,j)), j)
    else
        (m, k) ← (∞, 0)
        forall j ∈ S do
            (a, b) ← w((i,j)) + TSX(j, S \ {j})
            if a < m then
                (m, k) ← (a, b)
        return (m, k)
```

```
algorithm GenericBkt(v):
    if v is a solution then
        Return solution
    else
        foreach child u of v do
            if Promising(u) then
                GenericBkt(u)
```

Effectively implements a depth-first traversal of the state space of the given problem

- Possibly pruning the state space using PROMISING
- Improvement over the brute force
- However, the call to PROMISING may be missing for some problems
  - In this case backtracking offers no advantage run time-wise over brute force

- The graph m-colorability problem: Given an undirected graph G and an integer m, can the vertices of G be coloured with at most m colours such that no two adjacent vertices have the same colour
  - The smallest possible m is called the chromatic number of G
    - The maximum chromatic number of a planar graph is 4

```
algorithm COLOURS(⟨c_1, ..., c_i⟩, G = (V, E)):
    if i = n then return ⟨c_1, ..., c_n⟩
    else
        for c = 1 to m do
            if PROMISING(⟨c_1, ..., c_i, c⟩) then
                COLOURS(⟨c_1, ..., c_i, c⟩)
```

```
algorithm PROMISING(⟨c_1, ..., c_i⟩):
    j ← 1
    safe ← TRUE
    while j < i ∧ safe do
        if (i, j) ∈ E ∧ c_i = c_j then
            safe ← FALSE
        j ← j + 1
    return safe
```

BETTER BACKTRACKING FOR OPTIMIZATION PROBLEMS

- In optimization problems we can keep track of the best solution found so far and avoid expanding nodes if they would lead to a worse solution:

```
algorithm GENERICBKTOPT(v):
    if v is a solution then  Return solution
    else if VALUE(v) is better than bestsofar then  bestsofar ← VALUE(v)
    else if PROMISING(v) then
        foreach child u of v do  GENERICBKTOPT(u)
```

  - VALUE(v) is an upper/lower bound for all the solutions below v
  - *bestsofar* is a global variable maintained between different branches
  - PROMISING must reject nodes of less value than *bestsofar*

Better backtracking for optimization problems

- In optimization problems we can keep track of the best solution found so far and avoid expanding nodes if they would lead to a worse solution:

```
algorithm GenericBktOpt(v):
    if v is a solution then Return solution
    else if Value(v) is better than bestsofar then bestsofar ← Value(v)
    else if Promising(v) then
        foreach child u of v do GenericBktOpt(u)
```

  - Value(v) is an upper/lower bound for all the solutions below v
  - *bestsofar* is a global variable maintained between different branches
  - Promising must reject nodes of less value than *bestsofar*
- Case in point: 0/1 Knapsack revisited
  - The state space is binary (left child = pick item, right child = do not pick item)
  - Each state stores three values
    1. accumulated profit
    2. accomulated weight
    3. the upper bound Value() = the profit that can be made if the problem was fractional Knapsack
  - A node is not promising if either
    - The accumulated weight is larger than the capacity C, or
    - The upper bound is less than the maximum profit made so far

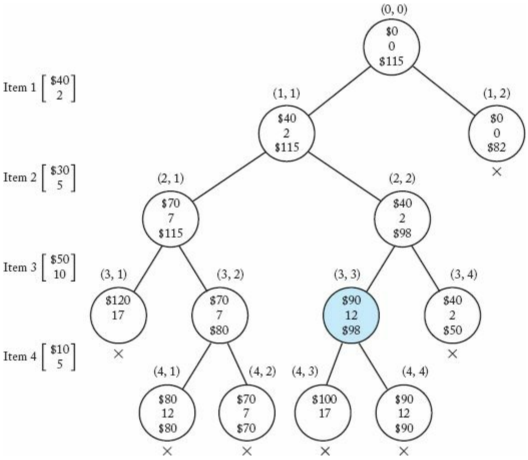
Obj:	1	2	3	4
p	40	30	50	10
w	2	5	10	5
p/w	20	6	5	2

$C = 16$

algorithm KNAPSACK():
- *bestsofar* ← 0
- **for** $i = 1$ **to** n **do** $result_i$ ← FALSE
- KNAPSACKREC(0, 0, 0)
- **return** (*bestsofar*, *bestset*)

algorithm KNAPSACKREC(i, *profit*, *weight*):
- **if** $weight \leq C \wedge profit > bestsofar$ **then**
  - *bestsofar* ← *profit*
  - *bestset* ← *result*
- **if** PROMISING(i) **then**
  - $result_{i+1}$ ← TRUE
  - KNAPSACKREC($i+1$, $profit+p_{i+1}$, $weight+w_{i+1}$)
  - $result_{i+1}$ ← FALSE
  - KNAPSACKREC($i+1$, *profit*, *weight*)

algorithm PROMISING(i):
- **if** $weight \geq C$ **then return** FALSE
- **else**
  - $j \leftarrow i + 1$
  - *bound* ← *profit*
  - W ← *weight*
  - **while** $j \leq n \wedge W + w_j \leq C$ **do**
    - $W \leftarrow W + w_j$
    - $bound \leftarrow bound + p_j$
    - $j \leftarrow j + 1$
  - **if** $k \leq n$ **then**
    - $bound \leftarrow bound + (C - W) \times p_j / w_j$
  - **return** $bound > profit$



- Similar to backtracking, but only for optimization problems
- Every time a state is considered its "value" is compared with the best solution candidate obtained so far
- Also changes the order of evaluation from depth first to
  - Breadth-first branch & bound
  - Best-first branch & bound where each node is associated a bound that denotes how "good" that node is

```
algorithm BRANCH&BOUND(v, bestsofar):
    open ← ⟨⟩
    ENQUEUE(v, open)
    bestsofar ← VALUE(v)
    while open ≠ ⟨⟩ do
        u ← DEQUEUE(open)
        foreach child u of v do
            if VALUE(u) > bestsofar then
                bestsofar ← VALUE(u)
            if BOUND(u) > bestsofar then
                ENQUEUE(u, open)
```

- *open* = queue → breadth-first branch & bound
- *open* = priority queue with key VALUE → best-first branch & bound

Obj:	1	2	3	4
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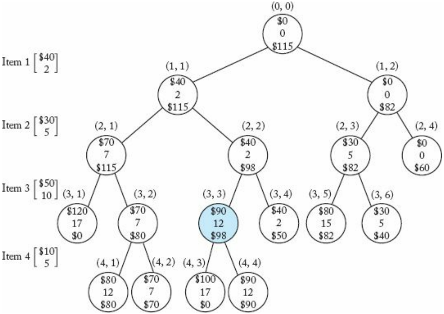
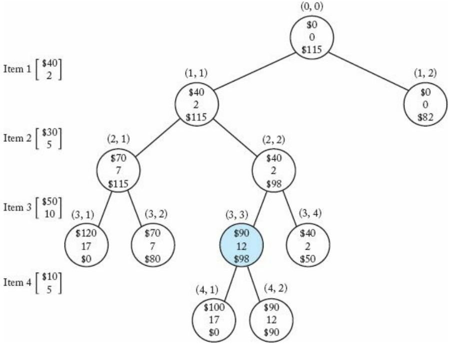
$C = 16$

- Breadth-first



When it is time to enqueue (2,3) its bound (82) is larger than *bestsofar* (70) so we enqueue and later expand. However, by the time we dequeue it *bestsofar* has already changed to 98.

- Best-first



Substantially smaller tree than in the case of breadth-first branch & bound.