Asymptotic Order Notations

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Time analysis

- Decide on the basic operation(s) then count
- Simple examples:
  1. **for** $i = 1$ **to** n **do**
     $sum \leftarrow sum + A_i$
     - With basic operation addition: $t(n) \approx n$
  2. // bubble sort
     for $i = 1$ **to** n **do**
     for $j = i + 1$ **to** n **do**
     if $A_j < A_i$ **then** $A_j \leftrightarrow A_i$
     - With basic operation swap: $t(n) \approx n^2$
  3. // matrix multiplication ($C = A \times B$)
     for $i = 1$ **to** n **do**
     for $j = i$ **to** n **do**
     $C_{i,j} \leftarrow 0$ **for** $k = 1$ **to** n **do**
     $C_{i,j} \leftarrow C_{i,j} + A_{i,k} \times B_{k,j}$
     - With multiplication as basic operation: $t(n) \approx n^3$

Time analysis (cont'd)

- **algorithm** SEQSEARCH(x, A, n):

```
i ← 1
found ← false
while ¬found ∧ i ≤ n do
    if A_i = x then
        found ← true
        return i
    else i ← i + 1
return −1
```

- Best case (x first): $t(n) \in O(1)$
- Worst case (x last or not in the sequence): $t(n) \in O(n)$
- Average case:
  - Assuming that x is in the list and equally likely to be in any position:
    $t(n) \approx \sum_{k=1}^{n} k \times 1/n = 1/n \sum_{k=1}^{n} k = (1/n)(n(n+1)/2) = (n+1)/2 \in O(n)$
  - Assuming x is in the list with probability p:
    $t(n) \approx p \times \sum_{k=1}^{n} k \times 1/n + (1-p) \times n = p \times (n+1)/2 + (1-p) \times n = pn/2 + p/2 + n - pn = n(1-p/2) + p/2 \in O(n)$

Anymptotic time analysis

- Behaviour of the algorithm (running time) for arbitrarily large input size
- Suppose:

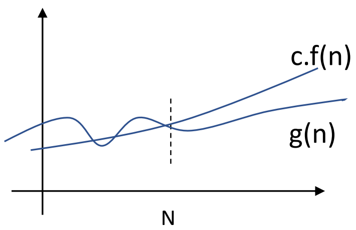
$$t_1(n) = 10n$$
$$t_2(n) = 0.1n^2$$
$$t_3(n) = 0.1n^2 + n + 100$$

n	$10n$		$0.1n^2$		$0.1n^2 + n + 100$
1	10		0.1		101.1
10	100		10		120
20	200	$>$	40	$<$	160
50	500		250		400
100	1000		1000		1200
1000	10,000	$<$	100,000	$\approx$	110,100
10,000	100,000		10,000,000		10,100,100

The big-Oh notation

- Asymptotic upper bound
- Several (equivalent) definitions:
  - $O(f(n))$ is the set of exactly all the functions that grow at most as fast as $f(n)$
    - The set of all functions with a smaller or equal rate of growth than $f(n)$
  - $g(n) \in O(f(n))$ iff $g(n)$ is bounded above by $f(n)$ except for a constant factor and a finite number of exceptions
  - Formally:

$$O(f(n)) = \{ \quad g(n) : \exists\, c > 0, N \geq 0 : \forall\, n \geq N : g(n) \leq c \times f(n) \quad \}$$



Big-Oh examples

- $5n^2 \in O(n^2)$?
  - $\iff \forall\, n \geq N : 5n^2 \leq c \times n^2$
  - True for $c = 5$ and $N = 0$
- $n \in O(n^2)$
  - $\iff \forall\, n \geq N : n \leq c \times n^2$
  - True for $c = 1$ and $N = 1$
- $n^2 \in O(n^2 + 10n)$
  - $\iff \forall\, n \geq N : n^2 \leq c \times (n^2 + 10n)$
  - Take $c = 1$
  - $\forall\, n \geq N : n^2 \leq n^2 + 10n$?
  - But $\forall\, n \geq N : 0 \leq 10n$
  - So true for $N = 0$
- $n^2 + 10n \in O(n^2)$
  - $\iff \forall\, n \geq N : n^2 + 10n \leq c \times n^2$
  - $\iff \forall\, n \geq N : n + 10 \leq c \times n$
  - But $n + 10 \leq n + 10n = 11n$
  - So true for $c = 11$ and $N = 1$

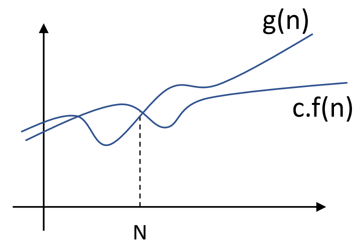
Big-Oh examples (cont'd)

- $n(n-1)/2 \in O(n^2)$?
  - $\iff \forall\, n \geq N : n(n-1)/2 \leq c \times n^2$
  - $\iff \forall\, n \geq N : (n-1)/2 \leq c \times n$
  - $\iff \forall\, n \geq N : n - 1 \leq c \times 2n$
  - Take $c = 0.5$:
  - $\iff \forall\, n \geq N : n - 1 \leq n \iff 1 \geq 0$
  - True for $c = 0.5$ and $N = 0$
- $2n + 3\log n \in O(n)$
  - $\iff \forall\, n \geq N : 2n + 3\log n \leq c \times n$
  - We know $\log n \leq n$ and so $2n + 3\log n \leq 2n + 3n = 5n$
  - So true for $c = 5$ and $N = 1$
- $n^2/2 \in O(n)$?
  - $\iff \forall\, n \geq N : n^2/2 \leq c \times n$
  - $\iff \forall\, n \geq N : n \leq 2c$
  - Cannot be true for unbounded n so false
- Can pick the constant c but cannot pick any particular n
  - In particular, the conclusion must be true for an unbounded n

The big-Ω notation

- Asymptotic lower bound
- Several (equivalent) definitions:
  - $\Omega(f(n))$ is a set that includes exactly all the functions that grow at least as fast as $f(n)$
    - The set of all functions with a larger or equal rate of growth than $f(n)$
  - $g(n) \in \Omega(f(n))$ iff $g(n)$ is bounded below by $f(n)$ except for a constant factor and a finite number of exceptions
  - Formally:

$$\Omega(f(n)) = \{ \quad g(n) : \exists\, c > 0, N \geq 0 : \forall\, n \geq N : g(n) \geq c \times f(n) \quad \}$$



Big-Ω examples

- $5n^2 \in \Omega(n^2)$?
  - $\iff \forall n \geq N : 5n^2 \geq c \times n^2$
  - True for $c = 5$ and $N = 0$
- $n^2 \in \Omega(n)$?
  - $\iff \forall n \geq N : n^2 \geq c \times n$
  - True for $c = 1$ and $N = 1$
- $5n–3 \in \Omega(n)$?
  - $\iff \forall n \geq N : 5n - 3 \geq c \times n$
  - Take $c = 1$
  - $\iff \forall n \geq N : 4n \geq 3$
  - True for $N = 1$
- $n(n-1)/2 \in \Omega(n^2)$?
  - $\iff \forall n \geq N : n(n-1)/2 \geq c \times n^2$
  - $\iff \forall n \geq N : n - 1 \geq 2c \times n$
  - We know that $\forall n \geq 2 : n - 1 \geq n/2$ ($n - 1 \geq n/2 \Rightarrow 2n - 2 \geq n \Rightarrow n \geq 2$)
  - So true for $2c = 1/2$ that is, $c = 1/4$ and $N = 2$
- $2n \in \Omega(n^2/2)$?
  - $\iff \forall n \geq N : 2n \geq c \times n^2/2$
  - $\iff \forall n \geq N : 4 \geq cn$
  - Cannot be true for unbounded n so false

The Θ notation

- Asymptotic tight bound
- Several (equivalent) definitions:
  - $\Theta(f(n))$ is a set that includes exactly all the functions that grow as fast as $f(n)$
    - The set of all functions with an equal rate of growth to $f(n)$
  - $g(n) \in \Theta(f(n))$ iff $g(n)$ is bounded above and below by $f(n)$ except for a constant factor and a finite number of exceptions
  - Formally:

$$\Omega(f(n)) = \{ \quad g(n) : \exists c_1 > 0, c_2 > 0, N \geq 0 : \forall n \geq N : c_1 \times f(n) \leq g(n) \leq c_2 \times f(n) \quad \}$$

- Therefore $\Theta(f(n)) = O(f(n)) \cap \Omega(f(n))$
  - In order to prove that $g(n) \in \Theta(f(n))$ one can prove that $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$
  - Example: $n(n-1)/2 \in \Theta(n^2)$ since we already showed earlier that $n(n-1)/2 \in O(n^2)$ and $n(n-1)/2 \in \Omega(n^2)$

Two useful properties of Θ

Theorem (complexity of polynomials)

Let $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n^1 + a_0$ be any polynomial in n of degree k with $a_k > 0$. Then $p(n) \in \Theta(n^k)$.

- We show that $p(n) \in O(n^k)$ by choosing $c = (k+1) \times \max_{i=1}^k a_i$ and noting that $(\max_{i=1}^k a_i)n^k \geq a_j n^k \geq a_j n^j$ for all $0 \leq j \leq k$
- To show that $p(n) \in \Omega(n^k)$ we note that $p(n) = \frac{a_k}{2} n^k + \sum_{j=0}^{k-1} \left(\frac{a_k}{2k} n^k + a_j n^j\right)$, we also note that $\left(\frac{a_k}{2k} n^k + a_j n^j\right) \geq 0$ for large enough n ($n \geq \max_{j=0}^{k-1} -2ka_j/a_k$), so $p(n) \geq \frac{a_k}{2} n^k$, and so $c = a_k/2k$
- Examples:
  - $n(n-1)/2 \in \Theta(n^2)$ (also shown without the theorem earlier)
  - $2n^2 - 3n \in \Theta(n^2)$

Theorem (logarithm base change)

$\log_a n = \Theta(\log_b n) = \Theta(\log n)$ *for any* $a, b > 1$

More asymptotic complexity notations

- $o(f(n))$ is the set of exactly all the functions with a strictly smaller rate of growth than $f(n)$
  - Formally:

$$o(f(n)) = \{ \quad g(n) : \forall c > 0 : \exists N \geq 0 : \forall n \geq N : g(n) \leq c \times f(n) \quad \}$$

- $\omega(f(n))$ is the set of exactly all the functions with a strictly larger rate of growth than $f(n)$
  - Formally:

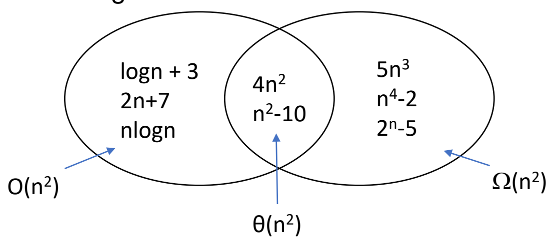
$$\omega(f(n)) = \{ \quad g(n) : \forall c > 0 : \exists N \geq 0 : \forall n \geq N : g(n) \geq c \times f(n) \quad \}$$

- Examples:
  - $n \in o(n^2) \iff \forall n \geq N : n \leq c \times n^2 \iff \forall n \geq N : 1 \leq cn \iff \forall n \geq N : n \geq 1/c$ (true for large enough n for any c)
  - $n \in o(4n) \iff \forall n \geq N : n \leq c \times 4n \iff 1 \leq 4c \iff c \geq 1/4$ (does not hold for all positive c, so false)
  - $n^2 \in \omega(n) \iff \forall n \geq N : n^2 \geq c \times n \iff \forall n \geq N : n \geq c$ (true for large enough n for any c)
  - $2n \in \omega(n) \iff \forall n \geq N : 2n \geq c \times n \iff 2 \geq c$ (does not hold for all positive c, so false)

Rules and properties

Order	Rate of growth of the two functions
O	$\leq$
Ω	$\geq$
Θ	$=$
o	$<$
ω	$>$

Venn diagram for orders:



	$g(n) \in \Theta(f(n))$	iff	$f(n) \in \Theta(g(n))$
	$g(n) \in O(f(n))$	iff	$f(n) \in \Omega(g(n))$
	$g(n) \in o(f(n))$	iff	$f(n) \in \omega(g(n))$
If	$g(n) \in o(f(n))$	then	$g(n) \in O(f(n))$
If	$g(n) \in \omega(f(n))$	then	$g(n) \in \Omega(f(n))$
	$g(n) \in \Theta(f(n))$	iff	$g(n) \in O(f(n)) \cap \Omega(f(n))$
If	$g(n) \in o(f(n))$	then	$g(n) \in O(f(n)) \setminus \Omega(f(n))$
If	$g(n) \in \omega(f(n))$	then	$g(n) \in \Omega(f(n)) \setminus O(f(n))$

All orders are transitive:
$\forall \mathbb{X} \in \{O, \Omega, \Theta, o, \omega\} : f(n) \in \mathbb{X}(g(n)) \wedge g(n) \in \mathbb{X}(h(n)) \Rightarrow f(n) \in \mathbb{X}(h(n))$

Rules and properties (cont'd)

- $O(f(n)) + O(g(n)) = O(f(n) + g(n))$
  - For example $O(n^2) + O(n) = O(n^2 + n)$ $(= O(n^2))$
  - Note: $O(n) + O(n) = O(n)$ (since $O(2n) = O(n)$)
  - Subtraction is trickier: $O(n) – O(n) = O(n)$!
- $O(f(n)) \times O(g(n)) = O(f(n) \times g(n))$
  - For example $O(n^2) \times O(n) = O(n^3)$
- $n \times O(f(n)) = O(n \times f(n))$
  - For example $\underbrace{O(n) + O(n) + ... + O(n)}_{n \text{ times}} = O(n^2)$
- $g(n) \in o(f(n))$ as long as $g(n)$ is in a set to the left of the set that includes $f(n)$ in the following list (with $k > j > 2$ and $b > a > 1$):

$$\Theta(\log n) \quad \Theta(n) \quad \Theta(n \log n) \quad \Theta(n^2) \quad \Theta(n^j) \quad \Theta(n^k) \quad \Theta(a^n) \quad \Theta(b^n) \quad \Theta(n!)$$

- Orders can be used in equations, with an implicit existential quantifier Examples:
  - $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means: there exists a function $f(n) \in \Theta(n)$ such that $2n^2 + 3n + 1 = 2n^2 + f(n)$
  - $2n^2 + \Theta(n) = \Theta(n^2)$ means: there exist functions $f(n) \in \Theta(n)$ and $g(n) \in \Theta(n^2)$ such that $2n^2 + f(n) = g(n^2)$

Using limits

- It is not always obvious how to prove a complexity claim
  - Especially when dealing with o and ω, which requires a proof for all constants
- We can then retort to limits:

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \begin{cases} c & \Rightarrow \quad g(n) \in \Theta(f(n)) \\ 0 & \Rightarrow \quad g(n) \in o(f(n)) \\ \infty & \Rightarrow \quad g(n) \in \omega(f(n)) \end{cases}$$

- When $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is not defined (e.g., ∞/∞) we can apply the l'Hôpital rule:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

  - Note in passing that all the complexity functions satisfy the prerequisites of the l'Hôpital rule

Examples of using limits

- $n^2 \log n \in O((n \log n)^2)$?
  - $\lim_{n\to\infty} \frac{n^2 \log n}{(n \log n)^2} = \lim_{n\to\infty} \frac{1}{\log n} = 0$
  - Therefore $n^2 \log n \in o((n \log n)^2) \Rightarrow n^2 \log n \in O((n \log n)^2)$
- $2^n \in \Theta(3^n)$?
  - $\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} (2/3)^n = 0$
  - Therefore $2^n \in o(3^n) \Rightarrow 2^n \notin \Theta(3^n)$
- $n \log n \in O(n^2)$?
  - $\lim_{n\to\infty} \frac{n \log n}{n^2} = \lim_{n\to\infty} \frac{\log n}{n} = \frac{\infty}{\infty}$, inconclusive
  - Apply l'Hôpital: $\lim_{n\to\infty} \frac{\log_b n}{n} = \lim_{n\to\infty} \frac{1/(n \ln b)}{1} = \frac{1}{n \ln b} = 0$
  - Therefore $n \log n \in o(n^2) \Rightarrow n \log n \in O(n^2)$
- $n^n \in ?(10^n)$
  - $\lim_{n\to\infty} \frac{n^n}{10^n} = \lim_{n\to\infty} (n/10)^n = \infty^\infty = \infty$
  - Therefore $n^n \in \omega(10^n)$ and also $n^n \in \Omega(10^n)$

Examples of using limits (cont'd)

- $8^{\log_2 n} \in ?(2^n)$
  - Useful properties: $\log_b a = \frac{\log_x a}{\log_x b}$ and $\log_a b = \frac{1}{\log_b a}$
  - We have $\log_2 n = \frac{\log_8 n}{\log_8 2} = \log_8 n \log_2 8 = 3 \log_8 n$, so $8^{\log_2 n} = n^3$
  - $\lim_{n\to\infty} \frac{n^3}{n^2} = \lim_{n\to\infty} n = \infty$ so $8^{\log_2 n} \in \omega(2^n)$ (and $8^{\log_2 n} \in \Omega(2^n)$)

Corollary (of the logarithm base change)

For $a, b > 1$ $a^{\log_b(n)} = O(n^k)$ with $k = \lceil 1/\log_b a \rceil$