Divide and Conquer

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Divide and Conquer

Idea:

1. If the problem is small enough, then solve it
2. Otherwise:
   1. Divide the problem into two or more sub-problems
   2. Solve each sub-problem recursively
   3. Combine the solutions to the sub-problems to obtain a solution to the original problem

Example:

```
algorithm MergeSort(S, l, h):
  if l < h then
    m ← (l + h)/2                  // divide
    MergeSort(S, l, m)             // conquer
    MergeSort(S, m + 1, h)         // conquer
    Merge(S, l, m, h)              // combine
```

```
algorithm Merge(S, l, m, h):
  T ← ⟨⟩                   // merge placeholder
  i ← l                    // top of first half
  j ← m                    // top of second half
  k ← l                    // top of T
  while i ≤ m ∧ j ≤ h do
    if S_i < S_j then      // compare top
      T_k ← S_i            // smaller in T
      i ← i + 1            // advance top
    else
      T_k ← S_j            // smaller in T
      i ← j + 1            // advance top
    k ← k + 1
  while i ≤ m do           // flush first half
    T_k ← S_i
    i ← i + 1
    k ← k + 1
  while j ≤ h do           // flush second half
    T_k ← S_j
    j ← j + 1
    k ← k + 1
  for k = l to h do        // result back into S
    S_k ← T_k
```


MergeSort analysis

Lemma (correctness of Merge)

If $S_{l\ldots m}$ and $S_{m+1\ldots h}$ are sorted then at the end of Merge the sequence $T_{l\ldots h}$ contains a sorted permutation of $S_{l\ldots h}$

- Loop invariant (for all three loops): $T_{l\ldots k-1}$ is sorted and contains exactly all the $k-1$ smallest elements of $S_{l\ldots h}$
  - Proof by induction over k
- At the end of the loop $k = h + 1$ and so the invariant implies the desired properties of T

Theorem (correctness of MergeSort)

MergeSort *replaces any input sequence $S_{h..l}$ with a sorted permutation of that sequence*

- Proof by induction on $h - l$:
  - In the base case $h - l = 0$ MergeSort (correctly) does nothing
  - To sort $h - l$ values MergeSort sorts correctly $(h - l)/2$ values two times (inductive hypothesis) and then correctly merges the two sub-sequences (lemma), thus obtaining a sorted permutation of the original sequence

MergeSort analysis (cont'd)

- $T(n) = 2T(n/2) + n$, $T(1) = 1$ so $T(n) = \Theta(n \log n)$ → already known!

Theorem (comparison sorting lower bound)

The lower bound for comparison sort algorithms is $\Omega(n \log n)$

- We count comparisons using a decision tree
  - Internal node $S_{i,j}$ represents a comparison between S_i and S_j
  - The left [right] sub-tree represents all the decisions to be made provided that $S_i \le S_j$ [$S_i > S_j$]
  - Each leaf labeled with a different permutation of S
  - Following a path performs the sequence of comparison given by the sequence of nodes and produces the leaf permutation of S
- We have $n!$ permutations (leafs) so the minimum path from root to a leaf contains $\log(n!) = \Theta(n \log n)$ nodes
- So a sorting algorithm must perform $\Omega(n \log n)$ comparisons to differentiate between all the possible permutations

Corollary (optimality of MergeSort)

MergeSort *is optimal*

QuickSort

- Problem with MergeSort: require substantial extra space
- By contrast QuickSort is an in-place sorting algorithm

```
algorithm QuickSort(S, l, h):
    if l < h then
        Choose pivot S_x
        S_1 ↔ S_x
        p ← Partition(S, l, h)
        QuickSort(S, l, p − 1)
        QuickSort(S, p + 1, h)
```

```
algorithm Partition(S, l, h):      // ver. 1
    pivot ← S_l
    j ← l
    for i = l + 1 to h do
        if S_i < pivot then
            j ← j + 1
            S_i ↔ S_j
    S_l ↔ S_j
    return j
```

```
algorithm Partition(S, l, h):      // ver. 2
    pivot ← S_l
    i ← l
    j ← h + 1                      // start beyond ends
    repeat
        repeat i ← i + 1 until S_i > pivot:
        repeat j ← j − 1 until S_j < pivot:
        if i < j then S_i ↔ S_j
    until i > j:
    S_l ↔ S_j
    return j
```


Analysis of QuickSort

- Time complexity:
  - Best case: we always partition equally
    $T(n) = 2T(n/2) + n$, $T(1) = 1$ and so $T(n) = \Theta(n \log n)$
  - Worst case: one partition is always empty (when?)
    $T(n) = T(n-1) + n$, $T(1) = 1$ and so $T(n) = \Theta(n^2)$
  - Can mitigate (but not fix) the worst case by choosing the pivot randomly of the best out of k random values for a small constant k
- QuickSort is not stable
- Correctness of Partition:
  - Loop invariant for version 1: At the end of an iteration all values $S_{l+1\ldots j}$ are smaller than *pivot* and no value $S_{j+1\ldots i}$ is smaller than *pivot*
  - Can verify by induction over i
  - Invariant implies desired postcondition that everything in $S_{l\ldots p-1}$ is less than *pivot* and nothing in $S_{p+1\ldots h}$ is less than the pivot
  - Loop invariant for version 2: At the end of an iteration all values in $S_{l+1\ldots i}$ are smaller than the pivot and no values in $S_{j\ldots h}$ are smaller than the pivot
  - Can verify by induction over the iteration number
- Correctness of QuickSort: same as for MergeSort (induction over $h - l$)

Linear-time selection

- We use the QuickSort idea to find the k-th smallest value in a given array, without sorting the array:

```
algorithm QuickSelect(k, S, l, h):
    if l < h then
        Choose pivot S_x
        S_1 ↔ S_x
        p ← Partition(S, l, h)
        if k = p then return S_k
        else if k < p then QuickSelect(k, S, l, p − 1)
        else QuickSelect(k, S, p + 1, h)
```

- Correctness: just like for QuickSort
- Time complexity:
  - Best case: we always partition equally
    $T(n) = T(n/2) + n$, $T(1) = 1$ and so $T(n) = \Theta(n)$ (better than sorting)
  - Worst case: one partition is always empty
    $T(n) = T(n-1) + n$, $T(1) = 1$ and so $T(n) = \Theta(n^2)$

How to choose good pivots

```
algorithm MoMSelect(k, S, l, h):
    if h − l ≤ 25 then use brute force
    else
        m ← (h − l)/5
        for i = 1 to m do
            M_i ← MedianOfFive(S_{l+5i−4...l+5i})    // brute force
            // Note: M can and should be an in-place array (within S)
        mom ← MoMSelect(m/2, M, 1, m)
        S_1 ↔ S_mom
        p ← Partition(S, l, h)
        if k = p then return S_k
        else if k < p then MoMSelect(k, S, l, p − 1)
        else MoMSelect(k, S, p + 1, h)
```

- Obviously correct (why?)
- *mom* is larger [smaller] than about $(h - l)/10$ block-of-five medians
- Each block median is larger [smaller] than 2 other elements in its block
- So *mom* is larger [smaller] than $3(h - l)/10$ elements in S and so cannot be farther than $7(h - l)/10$ elements from the perfect pivot
- So $T(n) = T(n/5) + T(7n/10) + n \Rightarrow T(n) = 10 \times c \times n \Rightarrow T(n) = \Theta(n)$
  - Note in passing: $T(n) = T(n/3) + T(2n/3) + n \Rightarrow T(n) = \Theta(n \log n)$
- If QuickSort uses MoMSelect to choose pivot then it gets down to $O(n \log n)$ worst-case complexity (optimal)

Fast matrix multiplication

With A and B $n \times n$ matrices compute $C = A \times B$ such that $C_{i,j} = \sum_{k-1}^{n} A_{i,k} \times B_{k,j}$

- Straightforward algorithm of complexity $O(n^3)$
- Obvious lower bound $\Omega(n^2)$
- Divide and conquer approach:

$$\left(\begin{array}{c|c} A_{\leftarrow\uparrow} & A_{\rightarrow\uparrow} \\ \hline A_{\leftarrow\downarrow} & A_{\rightarrow\downarrow}\end{array}\right) \times \left(\begin{array}{c|c} B_{\leftarrow\uparrow} & B_{\rightarrow\uparrow} \\ \hline B_{\leftarrow\downarrow} & B_{\rightarrow\downarrow}\end{array}\right) = \left(\begin{array}{c|c} C_{\leftarrow\uparrow} & C_{\rightarrow\uparrow} \\ \hline C_{\leftarrow\downarrow} & C_{\rightarrow\downarrow}\end{array}\right)$$

```
algorithm MatrixMul(n, A, B):
  if n = 2 then return A × B (brute force)
  else
    Partition A into A←↑, A→↑, A←↓, A→↓
    Partition B into B←↑, B→↑, B←↓, B→↓
    C←↑ ← MatrixMul(n/2, A←↑, B←↑)+MatrixMul(n/2, A→↑, B←↓)
    C→↑ ← MatrixMul(n/2, A←↑, B→↑)+MatrixMul(n/2, A→↑, B→↓)
    C←↓ ← MatrixMul(n/2, A←↓, B←↑)+MatrixMul(n/2, A→↓, B←↓)
    C→↓ ← MatrixMul(n/2, A←↓, B→↑)+MatrixMul(n/2, A→↓, B→↓)
    Combine C←↑, C→↑, C←↓, C→↓ into C
    return C
```

- $T(n) = 8T(n/2) + n^2, T(2) = 8 \Rightarrow T(n) = O(n^3)$ (bummer!)

Fast matrix multiplication (cont'd)

- To improve complexity we try to compute $C_{\leftarrow\uparrow}, C_{\rightarrow\uparrow}, C_{\leftarrow\downarrow}, C_{\rightarrow\downarrow}$ using less than 8 matrix multiplication operations
- Strassen's definitions:

$$\begin{aligned} P &= (A_{\leftarrow\uparrow} + A_{\rightarrow\uparrow})(B_{\leftarrow\uparrow} + B_{\rightarrow\downarrow}) \\ Q &= (A_{\rightarrow\uparrow} + A_{\rightarrow\downarrow})B_{\leftarrow\uparrow} \\ R &= A_{\leftarrow\uparrow}(B_{\rightarrow\uparrow} - B_{\rightarrow\downarrow}) \\ S &= A_{\rightarrow\downarrow}(B_{\rightarrow\uparrow} - B_{\leftarrow\uparrow}) \\ T &= (A_{\leftarrow\uparrow} + A_{\rightarrow\uparrow})B_{\rightarrow\downarrow} \\ U &= (A_{\rightarrow\uparrow} - A_{\leftarrow\uparrow})(B_{\leftarrow\uparrow} + B_{\rightarrow\uparrow}) \\ V &= (A_{\rightarrow\uparrow} - A_{\rightarrow\downarrow})(B_{\rightarrow\uparrow} + B_{\rightarrow\downarrow}) \end{aligned} \quad \text{so} \quad \begin{aligned} C_{\leftarrow\uparrow} &= P + S - T + V \\ C_{\rightarrow\uparrow} &= R + T \\ C_{\rightarrow\uparrow} &= Q + S \\ C_{\rightarrow\downarrow} &= P + R - Q + U \end{aligned}$$

- Only 7 multiplication operations!
- $T(n) = 7T(n/2) + n^2, T(2) = 8 \Rightarrow T(n) = O(n^{log 7}) = O(n^{2.81})$
  - Subsequent algorithms were able to bring complexity down to $O(n^{2.373})$
- Trick used: split into fewer (but less obvious) sub-problems

Large integer multiplication

Manipulate big integers → represented by arrays of n digits

- Obvious lower bound for addition and multiplication: $\Omega(n)$
- The straightforward algorithms are optimal for addition ($O(n)$) but not necessarily for multiplication ($O(n^2)$)
- Divide and conquer approach:
  - Let u and v be two n-digit integers
  - Let $m = n/2$ and let $u = x \times 10^m + y$ and $v = w \times 10^m + z$
  - It follows that $u \times v = (x \times 10^m + y)(w \times 10^m + z) = xw \times 10^{2m} + (xz + yw) \times 10^m + yz$

```
algorithm IntMul(n, u, v):
  m ← n/2
  if u = 0 ∨ v = 0 then return 0
  else if n = 2 then return u × v
  else
    x ← u DIV 10^m      // most significant m digits
    y ← u REM 10^m      // least significant m digits
    w ← v DIV 10^m
    z ← v REM 10^m
    return IntMul(m, x, w) × 10^2m
           +(IntMul(m, x, z)
             +IntMul(m, y, w)) × 10^m
           +IntMul(m, y, z)
```

- Running time: $T(n) = 4T(n/2) + n$, $T(2) = 4$
- Complexity: $O(n^2)$

Large integer multiplication (cont'd)

- Improvement:
  - Let $p_1 = xw$, $p_2 = yz$, $p_3 = (x + y)(w + z)$
  - Then $p_3 - p_1 - p_2 = (x + y)(w + z) - xw - yz = xz + yw$
  - Then $p = (x \times 10^m + y)(w \times 10^m + z) = xw \times 10^{2m} + (xz + yw) \times 10^m + yz = p_1 10^{2m} + (p3 - p1 - p2)10^m + p_2$

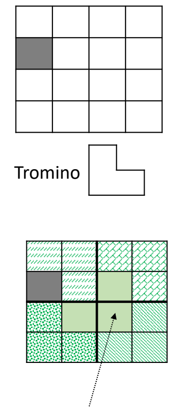
```
algorithm FastMul(n, u, v):
  m ← n/2
  if u = 0 ∨ v = 0 then return 0
  else if n = 2 then
    return u × v
  else
    x ← u DIV 10^m
    y ← u REM 10^m
    w ← v DIV 10^m
    z ← v REM 10^m
    p1 = FastMul(m, x, w)
    p2 = FastMul(m, y, z)
    p3 = FastMul(m, x + y, w + z)
    return p1 10^2m + (p3 − p1 − p2)10^m + p^2
```

- Running time: $T(n) = 3T(n/2) + n$, $T(2) = 4$
- Complexity: $O(n^{\log 3}) = O(n^{1.585})$

Tromino Tiling

Tile a bathroom floor ("board") with trominos without covering the drain (designated square on the board)

algorithm TILE(B, n, L): // B is the $n \times n$ board, L is the drain location
- **if** $n = 2$ **then**
  - Tile with one tromino without covering L
- **else**
  - Divide B into 4 $n/2 \times n/2$ sub-boards $B_1, \ldots, B_4$
  - Place a tromino to cover one square on each board that does not contain L
  - Let $L_1, \ldots L_4$ be the squares on each sub-board that are either covered or L
  - **for** $i = 1$ **to** 4 **do**
    - TILE(B_i,$n/2$,L_i)



Running time/trominoes used:

- $T(n) = 4T(n/2) + 1$, $T(2) = 1$
- $T(n) = 1/3(n^2 - 1)$
- Much better than the trial and error approach

When NOT to use divide and conquer

- Divide and conquer does not work for everything
- The crux of the technique is the ability to divide a problem into-sub problems
- Therefore divide and conquer is not the right thing to do when:
  - The size of sub-problems is the same (or larger) than the size of the original problem
    - Example: initial version of matrix or integer multiplication
    - Dramatic example: computing Fibonacci numbers
  - When the process of splitting into sub-problems takes too much time
  - When the process of combining the sub-solutions takes too much time