Data Structures

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Data structures recap

- Stack (FILO): push, pop, empty – constant time
- Queue (FIFO): insert, delete, empty – constant time
- Heaps: implementation of priority queue
  - Operations: insert ($O(\log n)$), peek (highest priority, $O(1)$), delete (highest priority, $O(\log n)$)
  - Tree representation, with children values smaller (maxheap) or larger (minheap) than the vertex value (weakly sorted)
  - Most efficiently implemented using arrays
  - Efficient sorting (heapsort)

DATA STRUCTURES RECAP (CONT'D)

- Trees: simple connected graph, one vertex may be designated as root
  - For a graph T with n vertices the following statements are equivalent:
    - T is a tree
    - T is connected and acyclic
    - T is connected and has $n-1$ edges
    - T is acyclic and has $n-1$ edges
  - Concepts: parent, ancestor, child, descendant, sibling, leaf, internal note
- Binary tree: each node had at most two children (left and right)
  - In a binary tree of height h with n nodes we have $h \geq \log_2 n$ (or $n \leq 2^h$)
  - Binary tree traversals ($O(n)$ complexity):

```
algorithm PREORDER(T):
  if ¬EMPTY(T) then
    VISIT(T)
    PREORDER(LEFT(T))
    PREORDER(RIGHT(T))

algorithm INORDER(T):
  if ¬EMPTY(T) then
    INORDER(LEFT(T))
    VISIT(T)
    INORDER(RIGHT(T))

algorithm POSTORDER(T):
  if ¬EMPTY(T) then
    POSTORDER(LEFT(T))
    POSTORDER(RIGHT(T))
    VISIT(T)
```

- Binary search tree: the value in every vertex is larger than all the values in its left subtree and smaller than all the values if its right subtree
  - Operations: insert, delete, search ($O(n)$ worst case, $O(\log n)$ if the tree is balanced)
  - Inorder traversal yields sorted sequence

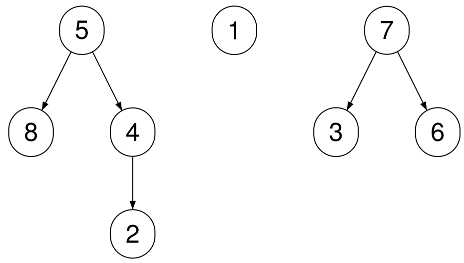
DISJOINT SETS

- Disjoint sets are non-empty, pairwise disjoint sets
  - Disjoint sets X_i, $1 \leq i \leq n$:
    $\forall 1 \leq i \leq n : X_i \neq \emptyset \quad \wedge \quad \forall 1 \leq i, j \leq n, i \neq j : X_i \cap X_j \neq \emptyset$
  - Each set has a member designated as the representative of that set
- Operations:
  - MAKESET(i): construct a set containing i as its sole element
  - FINDSET(i): return the representative of the set containing i
  - UNION(i, j): replaces the two sets containing i and j with their union; one of the two set representatives becomes the representative of the new set
- Representation: each set can be represented as a tree with the representative in the root
  - The tree does not have to be binary or balanced
- Implementation: disjoint sets over a domain D represented as an array *parent* indexed over D
  - $parent_i$ hold the parent of i in the tree representation, or i if i is the root

Disjoint sets (cont'd)

- Example: $\{2, 4, \underline{5}, 8\}$, $\{\underline{1}\}$, $\{3, 6, \underline{7}\}$

Tree representation:



Array implementation:

parent =

1	2	3	4	5	6	7	8
1	4	7	5	5	7	7	5

- A basic implementation:

algorithm MakeSet(i):
  $parent_i \leftarrow i$

algorithm FindSet(i):
  while $parent_i \neq i$ **do** $i \leftarrow parent_i$
  return i

algorithm Union(i, j):
  $x \leftarrow$ FindSet(i)
  $y \leftarrow$ FindSet(j)
  if $x \neq y$ **then** MergeTrees(x,y)

algorithm MergeTrees(i, j):
  $parent_i \leftarrow j$

- The tree representation can become very linear (depending on the sequence of calls to Union), so the running times are as follows:
  - MakeSet: $O(1)$
  - FindSet: $O(n)$
  - Union: $O(n)$ (since it calls FindSet)

Disjoint sets (cont'd)

- Weigthed union: To maintain a smaller tree height for the union we decide what tree gets the root based on the heights of the operands
- Maintain a height for each set (tree)
- During union the tree with the smallest height is attached to the root of the set with the larger height
  - The height stays the same
- When the two operands have the same height attach one to another (no matter which, but consistently)
  - The height increases by one
  - Overall for every two sets joined we have a height increase of at most one so no height in the tree is going over $\log n$
  - Better running times:
    - MakeSet: $O(1)$
    - FindSet: $O(\log n)$
    - Union: $O(\log n)$ (since it calls FindSet)

algorithm WUnion(i, j):
  $x \leftarrow$ FindSet(i)
  $y \leftarrow$ FindSet(j)
  if $x \neq y$ **then** WMergeTrees(x,y)

algorithm WMergeTrees(i, j):
  if $height_i > height_j$ **then** $parent_j \leftarrow i$
  else
    $parent_i \leftarrow j$
    if $height_i = height_j$ **then**
      $height_j \leftarrow height_j + 1$

DISJOINT SETS (CONT'D)

- Collapsing find: Each time we call FINDSET we collapse all the nodes we traverse so that they become connected directly to the root

  algorithm CFINDSET(i)**:**
  - **if** $i \neq parent_i$ **then** $parent_i \leftarrow$ CFINDSET($parent_i$)
  - **return** $parent_i$
- When using weighted union alone n MAKESET and m WUNION/FINDSET takes $O(n + m \log n)$ time
- When using weighted union and collapsing find n MAKESET and m WUNION/CFINDSET takes $O(n + m + \alpha(n, m))$ time where $\alpha(n, m)$ is a constant for all practical purposes

	MAKESET(i)	FIND(i)	UNION(i, j)	n MAKESET + m UNION/FIND
Basic impl.	$O(1)$	$O(n)$	$O(n)$	$O(n + nm)$
Weighted union	$O(1)$	$O(\log n)$	$O(\log n)$	$O(n + m \log n)$
Weighted union + collapsing find	$O(1)$	$O(\log n)$	$O(\log n)$	$\approx O(n + m)$

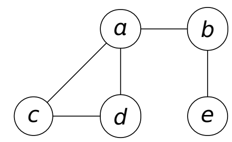
GRAPHS

- Directed graph (digraph): $G = (V, E)$ where V is a set of vertices and $E \subseteq V \times V$ is the set of edges
  - In a graphical representation edges are shown as arrows between vertices
- Undirected graph: A graph $G = (V, E)$ with the additional property that $(u, v) \in E$ iff $(v, u) \in E$
  - In a graphical representation edges are shown as lines between vertices
- Weighted graph: $G = (V, E, w)$ where (V, E) is a graph and $w : E \to \mathbb{R}$ associates a weight to each edge
  - In a graphical representation weights are shown as edge labels
- Concepts related to graphs:
  - adjacent vertices, degree, in degree, out degree
  - complement of $G = (V, E)$: $G' = (V, V \times V \setminus E)$
  - path, simple path, cycle, simple cycle
  - acyclic graph
  - length of the shortest path from u to v: DIST(u, v)
  - diameter of $G = (V, E)$: DIAM(G) $= \max\{$DIST$(u, v) : u, w \in V\}$
  - subgraph: a subset of edges along with all their vertices
  - induced subgraph: contains all the edges between its vertices
  - Hamiltonian cycle: cycle that contains each vertex exactly once
  - Euler cycle: cycle that contains each edge exactly once

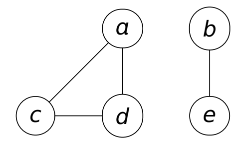
More types of graphs

- (Strongly) connected graph: graph that has a path between each pair of vertices
  - For a connected graph $G = (V, E)$ what is the minimum and the maximum $|E|$ (in terms of $|V|$)?
- Weakly connected graph: directed graph that is not connected but becomes connected if we transform it into an undirected graph
  - No concept of weak connectivity for undirected graphs (they are either connected or not)
- Clique or complete graph: $G = (V, V \times V)$
- Sparse vs dense graphs
- Bipartite graph: $G = (V_1 \uplus V_2, E)$ such that $E \subseteq V_1 \times V_2 \cup V_2 \times V_1$
  - Complete bipartite graph: $G = (V_1 \uplus V_2, V_1 \times V_2 \cup V_2 \times V_1)$

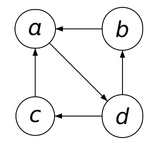
Connected:



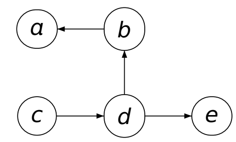
Unconnected:



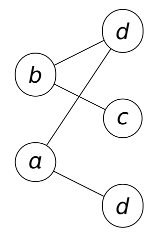
Strongly connected:



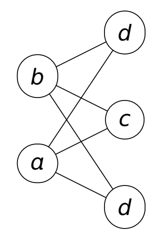
Weakly connected:



Bipartite:



Complete bipartite:



Graph representation

- Adjacency matrix
  - For $G = (V, E)$ establish an (arbitrary) order over V, such that we can consider $V = \{0, 1, \ldots, n\}$
  - Then G can be represented as the binary matrix $(G_{ij})_{0 \leq i,j \leq n}$ such that $G_{ij} = 1$ iff $(i, j) \in E$
  - For a weighted $G = (V, E, w)$ set $G_{ij} = w(i, j)$ if $(i, j) \in E$ and $G_{ij} = \infty$ otherwise

Undirected:

	a	b	c	d	e
a	0	1	1	1	0
b	1	0	0	0	1
c	1	0	0	1	0
d	1	0	1	0	0
e	0	1	0	0	0

Directed:

	a	b	c	d	e
a	0	0	0	1	0
b	1	0	0	0	0
c	1	0	0	1	0
d	0	0	0	0	0
e	0	1	0	0	0

Weighted:

	a	b	c	d	e
a	∞	5	2	1	∞
b	5	∞	∞	∞	8
c	2	∞	∞	2	∞
d	1	∞	2	∞	∞
e	∞	8	∞	∞	∞

- Adjacency list: For each vertex v use a list with exactly all the vertices u such that $(v, u) \in E$
  - Include the weights if it is a weighted graph

a	$\to b \to c \to d$
b	$\to a \to e$
c	$\to a \to d$
d	$\to a \to c$
e	$\to b$

a	$\to d$
b	$\to a$
c	$\to a \to d$
d	
e	$\to b$

a	$\to b, 5 \to c, 2 \to d, 1$
b	$\to a, 5 \to e, 8$
c	$\to a, 2 \to d, 2$
d	$\to a, 1 \to c, 2$
e	$\to b, 8$

- Time/space efficiency?

Graph traversal

algorithm TRAVERSE($G = (V, E)$):
- **foreach** $v \in V$ **do**
  - $visit_v \leftarrow$ false
- Let $v \in V$ such that $visit_v =$ false
- **if** v exists **then**
  - LISTTRAVERSE(v)

algorithm LISTTRAVERSE($v \in V$):
- $open \leftarrow \langle v \rangle$
- $visit_v \leftarrow$ true
- **while** $open \neq \langle\rangle$ **do**
  - $u \leftarrow$ HEAD($open$)
  - Output u
  - $new \leftarrow \langle x : (u, x) \in E \wedge \neg visited_x \rangle$
  - **foreach** $x \in new$ **do** $visit_x \leftarrow$ true
  - $open \leftarrow$ REST($open$) $\oplus$ new

Two different variants of $\oplus$ yield two different traversals:

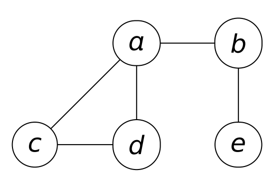
- Breath-first traversal: $L' \oplus L'' = L' + L''$
  - New vertices are added at the end and so *open* implements a queue
- Depth-first traversal: $L' \oplus L'' = L'' + L'$
  - New vertices are added at the beginning and so *open* implements a stack
  - Depth-first traversal can also be implemented recursively:

algorithm DFS($G = (V, E)$):
- **foreach** $v \in V$ **do** $visit_v \leftarrow$ false
- Let $v \in V$ such that $visit_v =$ false
- **if** v exists **then** RECDFS(v)

algorithm RECDFS($v \in V$):
- Output v
- $visit_v \leftarrow$ true
- **foreach** $(v, u) \in E \wedge \neg visit_u$ **do**
  - RECDFS(u)

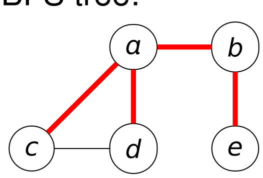
Graph traversal (cont'd)

- Any traversal of a graph G avoids all edges that would result in cycles
- Therefore it only expands (and thus defines) an acyclic subgraph of G = the traversal (DFS or BFS) tree

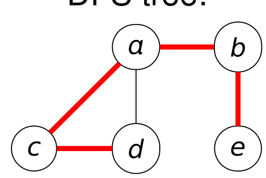


- Same traversal output starting from a: a, c, d, b, e
- Different traversal trees:

BFS tree:



DFS tree:



- Both algorithms run in time $O(n + m)$
- Space requirements however are vastly different

TOPOLOGICAL SORTING ON DIRECTED GRAPHS

- Given a graph $G = (V, E)$, obtain a linear ordering of V such that for every edge $(u, v) \in E$, u comes before v in the ordering

algorithm $\textsc{TSort}(G = (V, E))$**:**

> $order \leftarrow \langle\rangle$
> $S \leftarrow V$
> **while** $S \neq \emptyset$ **do**
> > Let $v \in S$ with in-degree 0
> > $order \leftarrow order + \langle v \rangle$
> > $E \leftarrow E \setminus \{(v, u) \in E\}$
> > $V \leftarrow V \setminus v$

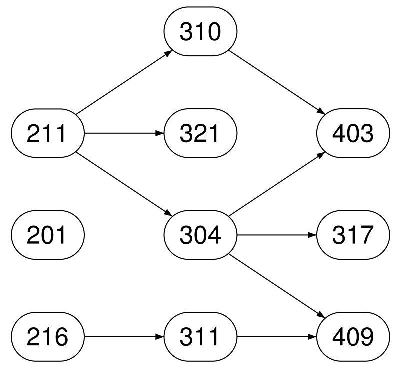
algorithm $\textsc{TSort'}(G = (V, E))$**:**

> $order \leftarrow \langle\rangle$
> $k \leftarrow n$
> **foreach** $v \in V$ **do** $visit_v \leftarrow$ false
> **while** $\exists\, v \in V : visit_v =$ false **do**
> > $\textsc{recTopo}(v)$

algorithm $\textsc{recTopo}(v \in V)$**:**

> $visit_v \leftarrow$ true
> **foreach** $(v, u) \in E \wedge \neg visit_u$ **do**
> > $\textsc{recTopo}(u)$
>
> $order_k \leftarrow v$
> $k \leftarrow k - 1$

- Many practical applications, e.g. sorting over a course prerequisite structure



Possible order:
$\langle 211, 310, 321, 201, 304, 403, 317, 216, 311, 409 \rangle$