# CS 455/555: Complexity theory

Stefan D. Bruda

Fall 2020

# TIME MATTERS

- For some $f : \mathbb{N} \to \mathbb{N}$, a Turing machine $M = (K, \Sigma, \Delta, s, \{h\})$ is $f$-time bounded iff for any $w \in \Sigma^*$: there is no configuration $C$ such that $(s, \#w\underline{\#}) \vdash_M^{f(|w|)+1} C$
- $M$ is polynomially (time) bounded iff $M$ is $p$-time bounded for some polynomial $p$
- $L \subseteq \Sigma^*$ is polynomially decidable iff there is a deterministic, polynomially bounded Turing machine that decides $L \Rightarrow$ complexity class $\mathcal{P}$
  - $\mathcal{P}$ is the class of exactly all the polynomially decidable languages
  - $\mathcal{P}$ is closed under complementation
  - There are recursive languages that are not in $\mathcal{P}$ (page 277)

$$E = \{\text{enc}(M)\#\text{enc}(w) : M \text{ accepts } w \text{ after at most } 2^{|w|} \text{ steps}\}$$

  - $\mathcal{P}$ (as well as subsequent complexity classes) are based on worst-case analysis

- For some $f : \mathbb{N} \to \mathbb{N}$, a Turing machine $M = (K, \Sigma, \Delta, s, \{h\})$ is *f*-time bounded iff for any $w \in \Sigma^*$: there is no configuration $C$ such that $(s, \#w\underline{\#}) \vdash_M^{f(|w|)+1} C$
- $M$ is polynomially (time) bounded iff $M$ is $p$-time bounded for some polynomial $p$
- $L \subseteq \Sigma^*$ is polynomially decidable iff there is a deterministic, polynomially bounded Turing machine that decides $L \Rightarrow$ complexity class $\mathcal{P}$
  - $\mathcal{P}$ is the class of exactly all the polynomially decidable languages
  - $\mathcal{P}$ is closed under complementation
  - There are recursive languages that are not in $\mathcal{P}$ (page 277)

$$E = \{\text{enc}(M)\#\text{enc}(w) : M \text{ accepts } w \text{ after at most } 2^{|w|} \text{ steps}\}$$

  - $\mathcal{P}$ (as well as subsequent complexity classes) are based on worst-case analysis
- Complexity class $\mathcal{NP}$: the class of exactly all the languages decided by nondeterministic, polynomially bounded Turing machines

# TIME MATTERS

- For some $f : \mathbb{N} \to \mathbb{N}$, a Turing machine $M = (K, \Sigma, \Delta, s, \{h\})$ is $f$-time bounded iff for any $w \in \Sigma^*$: there is no configuration $C$ such that $(s, \#w\underline{\#}) \vdash_M^{f(|w|)+1} C$
- $M$ is polynomially (time) bounded iff $M$ is $p$-time bounded for some polynomial $p$
- $L \subseteq \Sigma^*$ is polynomially decidable iff there is a deterministic, polynomially bounded Turing machine that decides $L \Rightarrow$ complexity class $\mathcal{P}$
  - $\mathcal{P}$ is the class of exactly all the polynomially decidable languages
  - $\mathcal{P}$ is closed under complementation
  - There are recursive languages that are not in $\mathcal{P}$ (page 277)

$$E = \{\text{enc}(M)\#\text{enc}(w) : M \text{ accepts } w \text{ after at most } 2^{|w|} \text{ steps}\}$$

  - $\mathcal{P}$ (as well as subsequent complexity classes) are based on worst-case analysis
- Complexity class $\mathcal{NP}$: the class of exactly all the languages decided by nondeterministic, polynomially bounded Turing machines
- Complexity class $\mathcal{EXP}$: exactly all the languages decided by exponentially-bounded, deterministic Turing machines
- $\mathcal{P} \subseteq \mathcal{NP} \subseteq \mathcal{EXP}$

# Alternative definition of $\mathcal{NP}$: certificates

- $L \in \Sigma^*; \Sigma^*$ is polynomially balanced iff there exists a polynomial $p$ such that $\forall\, x; y \in L : |y| \leqslant p(|x|)$
- $L \in \mathcal{NP}$ iff there exists a polynomially balanced language $L'$ such that
  1. $L' \in \mathcal{P}$, and
  2. $L = \{x \in \Sigma^* : \exists\, y \in \Sigma^* : x; y \in L'\}$
- $L'$ is the language of succinct certificates for $L$ (every $x \in L$ has a succinct certificate $y$)
- An $\mathcal{NP}$ problem has solutions that are easy to check

# Languages? Problems?

- Given some computational problem that requires certain resource (time) bounds to solve, it is generally easy to find a language that requires the same resource bounds to decide
  - Sometime (but not always) finding an algorithm for deciding the language immediately implies an algorithm for solving the problem
- Traveling salesman (TSP): Given $n \geqslant 2$, a matrix $(d_{ij})_{1 \leqslant i,j \leqslant n}$ with $d_{ij} > 0$ and $d_{ii} = 0$, find a permutation $\pi$ of $\{1, 2, \ldots, n\}$ such that $c(\pi)$, the cost of $\pi$ is minimal, where $c(\pi) = d_{\pi_1 \pi_2} + d_{\pi_2 \pi_3} + \cdots + d_{\pi_{n-1} \pi_n} + d_{\pi_n \pi_1}$
  - TSP the language (take 1): $\{((d_{ij})_{1 \leqslant i,j \leqslant n}, B) : n \geqslant 2, B \geqslant 0$, there exists a permutation $\pi$ such that $c(\pi) \leqslant B\}$
  - TSP the language (take 2), or the Hamiltonian graphs: Exactly all the graphs that have a (Hamiltonian) cycle that goes through all the vertices exactly once
  - Note: A cycle that uses all the edges exactly once is Eulerian; a graph $G$ is Eulerian iff
    1. There is a path between any two vertices that are not isolated, and
    2. Every vertex has an in-degree equal to its out-degree

- Clique: Given an undirected graph $G = (V, E)$, find the maximal set $C \subseteq V$ such that $\forall\, v_i, v_j \in C : (v_i, v_j) \in E$ ($C$ is a clique of $G$)
  - Clique, the language: $\{(G = (V, E), K) : K \geqslant 2 : \text{there exists a clique } C \text{ of } V \text{ such that } |C| \geqslant K\}$

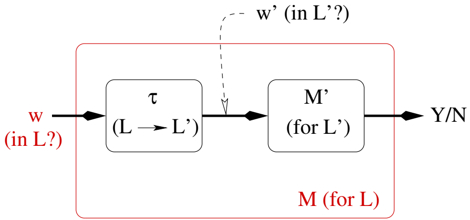
- Clique: Given an undirected graph $G = (V, E)$, find the maximal set $C \subseteq V$ such that $\forall\, v_i, v_j \in C : (v_i, v_j) \in E$ ($C$ is a clique of $G$)
  - Clique, the language: $\{(G = (V, E), K) : K \geqslant 2$ : there exists a clique $C$ of $V$ such that $|C| \geqslant K\}$
- SAT: Fix a set of variables $X = \{x_1, x_2, \ldots, x_n\}$ and let $\overline{X} = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$
  - An element of $X \cup \overline{X}$ is called a literal
  - A formula (or set/conjunction of clauses) is $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$ where $\alpha_i = x_{a_1} \vee x_{a_2} \vee \cdots \vee x_{a_k}$, $1 \leqslant i \leqslant m$, and $x_{a_i} \in X \cup \overline{X}$
  - An interpretation (or truth assignment) is a function $I : X \rightarrow \{\top, \bot\}$
  - A formula $F$ is satisfiable iff there exists an interpretation under which $F$ evaluates to $\top$.
  - SAT $= \{F : F$ is satisfiable $\}$
- 2-SAT, 3-SAT are variants of SAT (with the number of literals in every clause restricted to a maximum of 2 and 3, respectively)

## Theorem

$2\text{-}SAT \in \mathcal{P}$

- Algorithm **purge**$(F, x_i \in X)$: Erase from $F$ $\overline{x_i}$, erase from $F$ all the clauses that contain $x_i$
- Algorithm **satisfy**$(F, X)$:
  1. For every singleton clause $x_i$: Set $I(x_i) = \top$, **purge**$(F, x_i)$
  2. For every singleton clause $\overline{x_i}$: Set $I(x_i) = \bot$, **purge**$(F, \overline{x_i})$
  3. If we have an empty clause then report $F$ as unsatisfiable and stop
  4. Pick $x_i \in X$, set $X$ to $X \setminus \{x_i\}$, and copy $F$ into $F'$
  5. Set $I(x_i) = \top$, **purge**$(F, x_i)$
  6. If we have an empty clause, then
     1. Set $I(x_i) = \bot$, **purge**$(F', \overline{x_i})$
     2. If we have an empty clause then report $F$ as unsatisfiable and stop
     3. Set $F$ to $F'$
  7. If $x = \emptyset$ then report $F$ as satisfiable and stop, otherwise repeat from Step 4

- The general idea of reductions:




- Reductions can be used in proofs by contradiction:
  - If $L$ does not have property $\mathbb{P}$ and reduction $\tau$ from $L$ to $L'$ preserves $\mathbb{P}$
  - Then $L'$ does not have $\mathbb{P}$
- Example: Turing reductions and undecidable problems

# Polynomial reductions

- A function $f : \Sigma^* \to \Sigma^*$ is polynomially computable iff there exists a polynomially time bounded, deterministic Turing machine that computes it
- Let $L_1, L_2 \in \Sigma^*$; the function $\tau : \Sigma^* \to \Sigma^*$ is a polynomial reduction if it is polynomially computable, and $\forall\, x \in \Sigma^* : x \in L_1$ iff $\tau(x) \in L_2$
- Polynomial reductions show that a problem is not harder to solve than another within a polynomial-time factor

**Lemma**

*$L_1$ is polynomially reducible to $L_2$ and $L_2 \in \mathcal{P}$ implies $L_1 \in \mathcal{P}$*

**Theorem**

*Polynomial reductions are closed under (functional) composition*

- Direct, constructive proof

- A problem $L$ is $\mathcal{NP}$-hard iff for every language $L' \in \mathcal{NP}$ there exists a polynomial reduction from $L'$ to $L$
- A problem $L$ is $\mathcal{NP}$-complete iff $L$ is $\mathcal{NP}$-hard and $L \in \mathcal{NP}$

**Theorem**

*Let $L$ be some $\mathcal{NP}$-complete problem; then $\mathcal{P} = \mathcal{NP}$ iff $L \in \mathcal{P}$*

- $\Rightarrow$: $L$ is $\mathcal{NP}$-complete, so $L \in \mathcal{NP}$; however, $\mathcal{P} = \mathcal{NP}$ and so $L \in \mathcal{P}$
- $\Leftarrow$: $L \in \mathcal{P}$, so $L$ is decided by a polynomially time bounded deterministic machine $M$
  - For any $L' \in \mathcal{NP}$ we have a polynomial reduction $\tau$ from $L$ to $L'$, decided by a polynomially time bounded, deterministic machine $M_\tau$
  - Then $L'$ is decided by the deterministic, polynomially time bounded machine $M_\tau M$

- Reduction from Hamiltonian cycle to SAT
  - Graph $G$ given as adjacency matrix: $G = V \times V$, $V = \{1, 2, \ldots, n\}$
  - $G$ has a Hamiltonian cycle iff $\tau(G)$ is satisfiable
- Variables: $x_{ij}$, $1 \leqslant i, j \leqslant n$; $x_{ij} = \top$ iff vertex $i$ is number $j$ in the Hamiltonian cycle
- Clauses: need to specify that $x_{ij}$ represent a permutation (or bijection) over $V$; need then to specify that all the vertices in the cycle are actually connected
  1. at least one vertex is number $j$
  2. no vertex can be in two places at once
  3. every vertex must be in the cycle
  4. a place in the cycle can only have one vertex

  5. The permutation given by $x_{ij}$ is a Hamiltonian cycle

# Reduction example

- Reduction from Hamiltonian cycle to SAT
  - Graph $G$ given as adjacency matrix: $G = V \times V$, $V = \{1, 2, \ldots, n\}$
  - $G$ has a Hamiltonian cycle iff $\tau(G)$ is satisfiable
- Variables: $x_{ij}$, $1 \leqslant i, j \leqslant n$; $x_{ij} = \top$ iff vertex $i$ is number $j$ in the Hamiltonian cycle
- Clauses: need to specify that $x_{ij}$ represent a permutation (or bijection) over $V$; need then to specify that all the vertices in the cycle are actually connected
  1. at least one vertex is number $j$ $\quad \forall\, 1 \leqslant j \leqslant n : x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}$
  2. no vertex can be in two places at once
  3. every vertex must be in the cycle
  4. a place in the cycle can only have one vertex

  5. The permutation given by $x_{ij}$ is a Hamiltonian cycle

# Reduction example

- Reduction from Hamiltonian cycle to SAT
  - Graph $G$ given as adjacency matrix: $G = V \times V$, $V = \{1, 2, \ldots, n\}$
  - $G$ has a Hamiltonian cycle iff $\tau(G)$ is satisfiable
- Variables: $x_{ij}$, $1 \leqslant i, j \leqslant n$; $x_{ij} = \top$ iff vertex $i$ is number $j$ in the Hamiltonian cycle
- Clauses: need to specify that $x_{ij}$ represent a permutation (or bijection) over $V$; need then to specify that all the vertices in the cycle are actually connected
  1. at least one vertex is number $j$ $\quad \forall\, 1 \leqslant j \leqslant n : x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}$
  2. no vertex can be in two places at once $\quad \forall\, 1 \leqslant i, j, k \leqslant n, j \neq k : \overline{x_{ij}} \vee \overline{x_{ik}}$
  3. every vertex must be in the cycle
  4. a place in the cycle can only have one vertex

  5. The permutation given by $x_{ij}$ is a Hamiltonian cycle

- Reduction from Hamiltonian cycle to SAT
  - Graph $G$ given as adjacency matrix: $G = V \times V$, $V = \{1, 2, \ldots, n\}$
  - $G$ has a Hamiltonian cycle iff $\tau(G)$ is satisfiable
- Variables: $x_{ij}$, $1 \leqslant i, j \leqslant n$; $x_{ij} = \top$ iff vertex $i$ is number $j$ in the Hamiltonian cycle
- Clauses: need to specify that $x_{ij}$ represent a permutation (or bijection) over $V$; need then to specify that all the vertices in the cycle are actually connected
  1. at least one vertex is number $j$ $\quad \forall\, 1 \leqslant j \leqslant n : x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}$
  2. no vertex can be in two places at once $\quad \forall\, 1 \leqslant i, j, k \leqslant n, j \neq k : \overline{x_{ij}} \vee \overline{x_{ik}}$
  3. every vertex must be in the cycle $\quad \forall\, 1 \leqslant i \leqslant n : x_{i1} \vee x_{i2} \vee \cdots \vee x_{in}$
  4. a place in the cycle can only have one vertex

  5. The permutation given by $x_{ij}$ is a Hamiltonian cycle

- Reduction from Hamiltonian cycle to SAT
  - Graph $G$ given as adjacency matrix: $G = V \times V$, $V = \{1, 2, \ldots, n\}$
  - $G$ has a Hamiltonian cycle iff $\tau(G)$ is satisfiable
- Variables: $x_{ij}$, $1 \leqslant i, j \leqslant n$; $x_{ij} = \top$ iff vertex $i$ is number $j$ in the Hamiltonian cycle
- Clauses: need to specify that $x_{ij}$ represent a permutation (or bijection) over $V$; need then to specify that all the vertices in the cycle are actually connected
  1. at least one vertex is number $j$ $\quad \forall\, 1 \leqslant j \leqslant n : x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}$
  2. no vertex can be in two places at once $\quad \forall\, 1 \leqslant i, j, k \leqslant n, j \neq k : \overline{x_{ij}} \vee \overline{x_{ik}}$
  3. every vertex must be in the cycle $\quad \forall\, 1 \leqslant i \leqslant n : x_{i1} \vee x_{i2} \vee \cdots \vee x_{in}$
  4. a place in the cycle can only have one vertex
     $\forall\, 1 \leqslant i, j, k \leqslant n, i \neq k : \overline{x_{ij}} \vee \overline{x_{kj}}$
  5. The permutation given by $x_{ij}$ is a Hamiltonian cycle

# Reduction example

- Reduction from Hamiltonian cycle to SAT
  - Graph $G$ given as adjacency matrix: $G = V \times V$, $V = \{1, 2, \ldots, n\}$
  - $G$ has a Hamiltonian cycle iff $\tau(G)$ is satisfiable
- Variables: $x_{ij}$, $1 \leqslant i, j \leqslant n$; $x_{ij} = \top$ iff vertex $i$ is number $j$ in the Hamiltonian cycle
- Clauses: need to specify that $x_{ij}$ represent a permutation (or bijection) over $V$; need then to specify that all the vertices in the cycle are actually connected
  1. at least one vertex is number $j$ $\quad \forall\, 1 \leqslant j \leqslant n : x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}$
  2. no vertex can be in two places at once $\quad \forall\, 1 \leqslant i, j, k \leqslant n, j \neq k : \overline{x_{ij}} \vee \overline{x_{ik}}$
  3. every vertex must be in the cycle $\quad \forall\, 1 \leqslant i \leqslant n : x_{i1} \vee x_{i2} \vee \cdots \vee x_{in}$
  4. a place in the cycle can only have one vertex
     $\forall\, 1 \leqslant i, j, k \leqslant n, i \neq k : \overline{x_{ij}} \vee \overline{x_{kj}}$
  5. The permutation given by $x_{ij}$ is a Hamiltonian cycle $\quad$ For all $i$ and $k$ such that $(i, k) \notin G$ and assuming that $x_{kn+1} = x_{k1}$, we add $\overline{x_{ij}} \vee \overline{x_{kj+1}}$

# Reduction example (cont'd)

- We have $O(n^3)$ clauses with at most $O(n)$ literals each
- Each clause may depend on $G$ and $n$ but nothing else
- The whole set is clearly polynomially computable, as desired
- Remains to prove that $G$ has a Hamiltonian cycle iff $\tau(G)$ is satisfiable
  - Suppose that some interpretation $I$ satisfies $\tau(G)$
  - Then for each $i$ exactly one $I(x_{ij})$ is $\top$ and for each $j$ exactly one $I(x_{ij})$ is $\top$ (because of 1-4)
  - This goes both ways
  - if
    - $\overline{x_{ij}} \vee \overline{x_{kj+1}}$ is true whenever $(i, j) \notin G$
    - Whenever $i = \pi_j$ and $k = \pi_{j+1}$ we have $I(x_{ij}) = \top$ and $I(x_{kj+1}) = \top$
    - Therefore the clause $\overline{x_{ij}} \vee \overline{x_{kj+1}}$ if false, so (i,k) must be an edge in $G$
  - only if
    - Let $\pi$ be a Hamiltonian cycle
    - We then set $I(x_{ij}) = \top$ iff $j = \pi_i$, which makes $\tau(G)$ true

# $\mathcal{NP}$-COMPLETENESS THEORY IN A NUTSHELL

- Are there $\mathcal{NP}$-complete problems at all?
  - Yes, SAT is one (cf. Stephen Cook, 1971)
- The first is the hard one: we have to show that every problem in $\mathcal{NP}$ reduces to our problem
- Then in order to find other $\mathcal{NP}$-complete problems all we need to do is to find problems such that some problem already known to be $\mathcal{NP}$-complete reduces to them
  - This works because polynomial reductions are closed under composition = are transitive
- Then it is apparently easy to use the theorem stated earlier:

  Let $L$ be some $\mathcal{NP}$-complete problem; then $\mathcal{P} = \mathcal{NP}$ iff $L \in \mathcal{P}$

- Tiling system: $\mathcal{D} = (D, d_0, H, V)$
  - $D$ is a finite set of tiles
  - $d_0 \in D$ is the initial corner tile
  - $H, V \in D \times D$ are the horizontal and vertical tiling restrictions
- Tiling: $f : \mathbb{N}_s \times \mathbb{N}_s \to D$ such that
  - $f(0,0) = d_0$
  - $\forall\, 0 \leqslant m < s, 0 \leqslant n < s-1 : (f(m,n), f(m,n+1)) \in V$
  - $\forall\, 0 \leqslant m < s-1, 0 \leqslant n < s : (f(m,n), f(m+1,n)) \in H$
- The bounded tiling problem:
  - Given a tiling system $\mathcal{D}$, a positive integer $s$ and an initial tiling $f_0 : \mathbb{N}_s \to D$
  - Find whether there exists a tiling function $f$ that extends $f_0$

# BOUNDED TILING IS $\mathcal{NP}$-COMPLETE

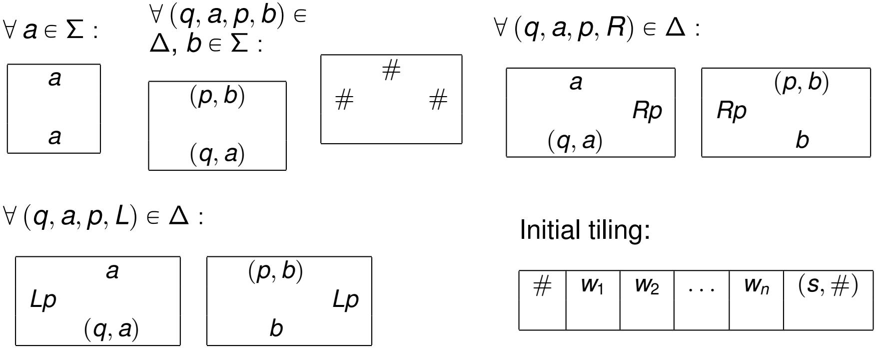
- We need to find reductions from all problems in $\mathcal{NP}$ to bounded tiling

# Bounded tiling is $\mathcal{NP}$-complete

- We need to find reductions from all problems in $\mathcal{NP}$ to bounded tiling
  - The only thing in common to all the $\mathcal{NP}$ problems is that each of them is decided by a nondeterministic, polynomially bounded Turing machine
  - We therefore find a reduction from an arbitrary such a machine to bounded tiling

# Bounded tiling is $\mathcal{NP}$-complete

- We need to find reductions from all problems in $\mathcal{NP}$ to bounded tiling
  - The only thing in common to all the $\mathcal{NP}$ problems is that each of them is decided by a nondeterministic, polynomially bounded Turing machine
  - We therefore find a reduction from an arbitrary such a machine to bounded tiling
- We find a tiling system such that each row in the tiling corresponds to one configuration of the given Turing machine



# SAT IS $\mathcal{NP}$-COMPLETE

1. SAT $\in \mathcal{NP}$
   - We nondeterministically guess an interpretation and we check that the interpretation satisfies the formula
   - Both of these take linear time
2. SAT is $\mathcal{NP}$-hard
   - By reduction of bounded tiling to SAT
   - Consider variables $x_{nmd}$ standing for "tile $d$ is at position $(n, m)$ in the tiling"
   - Construct clauses such that $x_{nmd} = \top$ iff $f(m, n) = d$
   - We first specify that we have a function:
     - each position has at least one tile: $\forall\, 0 \leqslant m, n \leqslant s : x_{mnd_1} \vee x_{mnd_2} \vee \cdots$
     - no more than one tile in a given position: $\forall\, 0 \leqslant m, n \leqslant s, d \neq d' : \overline{x_{mnd}} \vee \overline{x_{mnd'}}$
   - Then we specify the restrictions $H$ and $V$:
     - $(d, d') \in D^2 \backslash H \Rightarrow \overline{x_{mnd}} \vee \overline{x_{m+1nd'}}$ $\qquad (d, d') \in D^2 \backslash V \Rightarrow \overline{x_{mnd}} \vee \overline{x_{mn+1d'}}$

- In fact 3-SAT is also $\mathcal{NP}$-complete

# Proof of $\mathcal{NP}$-completeness

- To show that a problem is $\mathcal{NP}$-complete we need to show that
- The problem is in $\mathcal{NP}$
  - Construct a Turing machine, or find succinct certificates
  - Usually quite straightforward
- The problem is $\mathcal{NP}$-hard
  - Exhibit a polynomial reduction from a known $\mathcal{NP}$-complete problem
  - Reduction can happen from any problem discussed in class and also from any problem discussed in Sections 7.2 and 7.3 (take those problems as solved exercises)
  - Make sure that you are comfortable with this way of thinking! There are numerous solved exercises to make you comfortable

# CLIQUE

- 3-SAT is $\mathcal{NP}$-complete

# CLIQUE

- 3-SAT is $\mathcal{NP}$-complete
  - Hint: any clause $x_1 \vee x_2 \vee \cdots x_n$ is logically equivalent with $(x_1 \vee x_2 \vee x_2') \wedge (\overline{x_2'} \vee x_3 \vee x_3') \wedge (\overline{x_3'} \vee x_4 \vee x_4') \wedge \cdots \wedge (\overline{x_{n-2}'} \vee x_{n-1} \vee x_n)$
- CLIQUE $= \{(G = (V, E), k) : k \geqslant 2$ : there exists a clique $C$ of $V$, $|C| = k\}$
  Membership in $\mathcal{NP}$ and 3-SAT being reducible to CLIQUE implies CLIQUE is $\mathcal{NP}$-complete

# CLIQUE

- 3-SAT is $\mathcal{NP}$-complete
  - Hint: any clause $x_1 \vee x_2 \vee \cdots x_n$ is logically equivalent with $(x_1 \vee x_2 \vee x_2') \wedge (\overline{x_2'} \vee x_3 \vee x_3') \wedge (\overline{x_3'} \vee x_4 \vee x_4') \wedge \cdots \wedge (\overline{x_{n-2}'} \vee x_{n-1} \vee x_n)$
- CLIQUE $= \{(G = (V, E), k) : k \geqslant 2 :$ there exists a clique $C$ of $V$, $|C| = k\}$
  Membership in $\mathcal{NP}$ and 3-SAT being reducible to CLIQUE implies CLIQUE is $\mathcal{NP}$-complete
  - Start from $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_k$, construct $G = (V, E)$
  - Start with $V = \varnothing$ and $E = \varnothing$
  - For each clause $C_r = l_1^r \vee l_2^r \vee l_3^r$ add vertices $v_1^r$, $v_2^r$, and $v_3^r$ to $V$
  - Add $(v_i^r, v_j^s)$ to E whenever $r \neq s$ and $l_i^r$ is not the negation of $l_j^s$ ($l_i^r$ is and $l_j^s$ are consistent)

# CLIQUE

- 3-SAT is $\mathcal{NP}$-complete
  - Hint: any clause $x_1 \vee x_2 \vee \cdots x_n$ is logically equivalent with $(x_1 \vee x_2 \vee x_2') \wedge (\overline{x_2'} \vee x_3 \vee x_3') \wedge (\overline{x_3'} \vee x_4 \vee x_4') \wedge \cdots \wedge (\overline{x_{n-2}'} \vee x_{n-1} \vee x_n)$
- CLIQUE $= \{(G = (V, E), k) : k \geqslant 2 :$ there exists a clique $C$ of $V$, $|C| = k\}$
  Membership in $\mathcal{NP}$ and 3-SAT being reducible to CLIQUE implies CLIQUE is $\mathcal{NP}$-complete
  - Start from $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_k$, construct $G = (V, E)$
  - Start with $V = \emptyset$ and $E = \emptyset$
  - For each clause $C_r = l_1^r \vee l_2^r \vee l_3^r$ add vertices $v_1^r$, $v_2^r$, and $v_3^r$ to $V$
  - Add $(v_i^r, v_j^s)$ to E whenever $r \neq s$ and $l_i^r$ is not the negation of $l_j^s$ ($l_i^r$ is and $l_j^s$ are consistent)
  - Suppose that $\phi$ is satisfiable; then:
    - The interpretation that makes $\phi$ true makes at least one literal $l_i^r$ per clause true
    - The vertex $v_i^r$ is connected to all the other vertices $v_j^s$ that make the other clauses true (these are all consistent with each other)
    - So the vertices $v_i^r$ form a clique (of size $k$)

- 3-SAT is $\mathcal{NP}$-complete
  - Hint: any clause $x_1 \vee x_2 \vee \cdots x_n$ is logically equivalent with $(x_1 \vee x_2 \vee x_2') \wedge (\overline{x_2'} \vee x_3 \vee x_3') \wedge (\overline{x_3'} \vee x_4 \vee x_4') \wedge \cdots \wedge (\overline{x_{n-2}'} \vee x_{n-1} \vee x_n)$
- CLIQUE $= \{(G = (V, E), k) : k \geqslant 2 :$ there exists a clique $C$ of $V$, $|C| = k\}$
  Membership in $\mathcal{NP}$ and 3-SAT being reducible to CLIQUE implies CLIQUE is $\mathcal{NP}$-complete
  - Start from $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_k$, construct $G = (V, E)$
  - Start with $V = \varnothing$ and $E = \varnothing$
  - For each clause $C_r = l_1^r \vee l_2^r \vee l_3^r$ add vertices $v_1^r$, $v_2^r$, and $v_3^r$ to $V$
  - Add $(v_i^r, v_j^s)$ to E whenever $r \neq s$ and $l_i^r$ is not the negation of $l_j^s$ ($l_i^r$ is and $l_j^s$ are consistent)
  - Suppose that $\phi$ is satisfiable; then:
    - The interpretation that makes $\phi$ true makes at least one literal $l_i^r$ per clause true
    - The vertex $v_i^r$ is connected to all the other vertices $v_j^s$ that make the other clauses true (these are all consistent with each other)
    - So the vertices $v_i^r$ form a clique (of size $k$)
  - Suppose that $G$ has a clique $C$ of size $k$; then:
    - $C$ contains exactly one vertex per clause
    - Assigning $\top$ to every literal $l_i^r$ for which $v_i^r \in C$ is possible (all are consistent with each other)
    - The assignment makes $\phi$ true so $\phi$ is satisfiable

- A vertex cover of $G = (V, E)$ is a set $V' \subseteq V$ such that $(u, v) \in E \Rightarrow u \in V' \vee v \in V'$
- VERTEX-COVER $= \{(G = (V, E), k) : G \text{ has a vertex cover of size } k\}$
  Membership in $\mathcal{NP}$ and CLIQUE being reducible to VERTEX-COVER implies VERTEX-COVER is $\mathcal{NP}$-complete

- A vertex cover of $G = (V, E)$ is a set $V' \subseteq V$ such that $(u, v) \in E \Rightarrow u \in V' \vee v \in V'$
- VERTEX-COVER $= \{(G = (V, E), k) : G$ has a vertex cover of size $k\}$
  Membership in $\mathcal{NP}$ and CLIQUE being reducible to VERTEX-COVER implies VERTEX-COVER is $\mathcal{NP}$-complete
  - Start from $(G = (V, E), k) \in$ CLIQUE
  - Compute $\overline{G} = (V, \overline{E})$ where $\overline{E} = (V \times V) \backslash E$ (the complement of $G$)
  - Then $(G, k) \in$ CLIQUE iff $(\overline{G}, |V| - k) \in$ VERTEX-COVER

- A vertex cover of $G = (V, E)$ is a set $V' \subseteq V$ such that $(u, v) \in E \Rightarrow u \in V' \vee v \in V'$
- Vertex-cover $= \{(G = (V, E), k) : G \text{ has a vertex cover of size } k\}$
  Membership in $\mathcal{NP}$ and Clique being reducible to Vertex-cover implies Vertex-cover is $\mathcal{NP}$-complete
  - Start from $(G = (V, E), k) \in$ Clique
  - Compute $\overline{G} = (V, \overline{E})$ where $\overline{E} = (V \times V) \setminus E$ (the complement of $G$)
  - Then $(G, k) \in$ Clique iff $(\overline{G}, |V| - k) \in$ Vertex-cover
  - Suppose that $G$ has a clique $C$, $|C| = k$; then:
    - $(u, v) \notin E$ means that $u$ and $v$ cannot be both in $C$
    - That is, $V \setminus C$ covers every edge $(u, v) \notin E$ that is, every vertex $(u, v) \in \overline{E}$
    - Therefore $V \setminus C$ is a vertex cover for $\overline{G}$ (of size $|V| - k$)

- A vertex cover of $G = (V, E)$ is a set $V' \subseteq V$ such that $(u, v) \in E \Rightarrow u \in V' \vee v \in V'$
- VERTEX-COVER $= \{(G = (V, E), k) : G$ has a vertex cover of size $k\}$
  Membership in $\mathcal{NP}$ and CLIQUE being reducible to VERTEX-COVER implies VERTEX-COVER is $\mathcal{NP}$-complete
  - Start from $(G = (V, E), k) \in$ CLIQUE
  - Compute $\overline{G} = (V, \overline{E})$ where $\overline{E} = (V \times V) \backslash E$ (the complement of $G$)
  - Then $(G, k) \in$ CLIQUE iff $(\overline{G}, |V| - k) \in$ VERTEX-COVER
  - Suppose that $G$ has a clique $C$, $|C| = k$; then:
    - $(u, v) \notin E$ means that $u$ and $v$ cannot be both in $C$
    - That is, $V \backslash C$ covers every edge $(u, v) \notin E$ that is, every vertex $(u, v) \in \overline{E}$
    - Therefore $V \backslash C$ is a vertex cover for $\overline{G}$ (of size $|V| - k$)
  - Suppose that $\overline{G}$ has a vertex cover $V'$ with $|V'| = |V| - k$; then:
    - $(u, v) \in \overline{E} \Rightarrow u \in V' \vee v \in V'$
    - Contrapositive: $u \notin V' \wedge v \notin V' \Rightarrow (u, v) \notin \overline{E}$
    - That is, $u \in V \backslash V' \wedge v \in V \backslash V' \Rightarrow (u, v) \in E$
    - So $V \backslash V'$ is a clique of $G$ (or size $k$)

# Other issues related to $\mathcal{P}$ and $\mathcal{NP}$

- co-$\mathcal{NP}$ is the complement of $\mathcal{NP}$ ($P \in$ co-$\mathcal{NP}$ iff $\overline{P} \in \mathcal{NP}$)
  - Thought to be different from $\mathcal{NP}$
  - $\mathcal{P} \subseteq$ co-$\mathcal{NP}$, $\mathcal{P} \subseteq \mathcal{NP}$
  - If $P \in$ co-$\mathcal{NP}$, $P \in \mathcal{NP}$, and $P \in \mathcal{P}$ then $P$ is suspected not to be $\mathcal{NP}$-complete
  - Example: the language of composite numbers (aka the integer factorization problem)
    - in $\mathcal{NP}$ and also in co-$\mathcal{NP}$
    - suspected outside $\mathcal{P}$
    - suspected outside $\mathcal{NP}$-complete
- co-$\mathcal{NP}$-complete problems also definable
  - integer factorization also suspected outside co-$\mathcal{NP}$-complete

- Several complexity classes:
  $\mathcal{L} \subseteq \mathcal{NL} \subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq \mathcal{PSPACE} = \mathcal{NPSPACE}$
  - $\mathcal{L}$ stands for logarithmic space and $\mathcal{NL}$ for nondeterministic logarithmic space
- The only thing known: $\mathcal{NL} \neq \mathcal{PSPACE}$
  - So at least one of the inclusions in between must be strict
  - But we do not know which ones are strict or not
- Each inclusion has its own completeness theory, so we have $\mathcal{P}$-complete and $\mathcal{PSPACE}$-complete problems
  - The reduction for each completeness theory comes from the inner class
    - Indeed, if we go higher then all problems in the given class become complete!
  - That is, $\mathcal{P}$-complete problems are defined in terms of $\mathcal{NL}$ reductions, whereas $\mathcal{PSPACE}$-complete problems are defined in terms of $\mathcal{NP}$ reductions