# CS 455/555: Context-free languages

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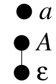
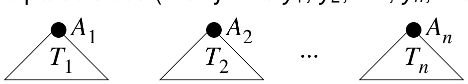
## Grammars

- A grammar is a tuple $G = (V, \Sigma, R, S)$, where
  - $V$ is an alphabet; $\Sigma \subseteq V$ is the alphabet of terminals
    - $V \setminus \Sigma$ called by contrast nonterminals
  - $S \in V \setminus \Sigma$ is the axiom (or the start symbol)
  - $R \subseteq V^*(V \setminus \Sigma)V^* \times V^*$ is the set of (rewriting) rules or productions
    - Common ways of expressing $(\alpha, \beta) \in R$ for a grammar $G$: $\alpha \to_G \beta$ (or just $\alpha \to \beta$), $\alpha \to \beta \in R$
- Context-free grammar: a grammar with $R \subseteq (V \setminus \Sigma) \times V^*$
  - (left) regular grammar: $R \subseteq (V \setminus \Sigma) \times (\Sigma(V \setminus \Sigma) \cup \{\varepsilon\})$
- $u \Rightarrow_G v$ iff $\exists x, y \in V^* : \exists A \in V \setminus \Sigma : u = xAy,\ v = xv'y,\ A \to_G v'$
- $\Rightarrow_G^*$ is the reflexive and transitive closure of $\Rightarrow_G$ (derivation)
- The language generated by grammar $G$: $\mathcal{L}(G) = \{w \in \Sigma^* : S \Rightarrow_G^* w\}$

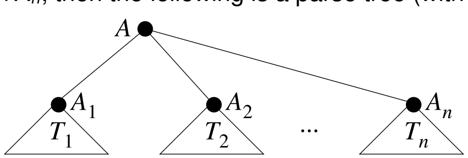


## Parse trees

- Tree with labelled nodes
- Yield: concatenation of leaves in inorder
- Definition:
  1. For every $a \in \Sigma$ the following is a parse tree (with yield $a$): 
  2. For every $A \to \varepsilon$ the following is a parse tree (with yield $\varepsilon$): 
  3. If the following are parse trees (with yields $y_1, y_2, \ldots, y_n$, respectively):



and $A \to A_1 A_2 \ldots A_n$, then the following is a parse tree (with yield $y_1 y_2 \ldots y_n$):





## Derivations and parse trees

- Every derivation starting from some nonterminal has an associated parse tree (rooted at the starting nonterminal)
- Two derivations are similar iff only the order of rule application varies
  - Can obtain one derivation from the other by repeatedly flipping consecutive rule applications
  - Two similar derivations have identical parse trees
  - Can always choose a "standard" derivations: leftmost ($A \overset{L}{\Rightarrow}{}^* w$) or rightmost ($A \overset{R}{\Rightarrow}{}^* w$)

**Theorem**

*The following four statements are equivalent:*
- *There exists a parse tree with root $A$ and yield $w$*
- $A \Rightarrow^* w$
- $A \overset{L}{\Rightarrow}{}^* w$
- $A \overset{R}{\Rightarrow}{}^* w$

- Ambiguity of a grammar: there exists a string that has two derivations that are not similar (i.e., two derivations with diferent parse trees)
  - Can be inherent or not

## Context-free and regular languages

- Languages generated by context-free grammars are called context-free

**Theorem**

*Exactly all the regular languages are generated by regular grammars (which are all context-free grammars)*

- Let $M = (K, \Sigma, \Delta, s, F)$ be some finite automaton
- We construct the grammar $G = (K \cup \Sigma, \Sigma, s, R)$ with

$$R = \{q \to ap : (q, a, p) \in \Delta\} \cup \{q \to \varepsilon : q \in F\}$$

**Corollary**

*All regular languages are context-free*

- However, there are more context-free than regular languages

$$S \to aSb \quad S \to \varepsilon$$

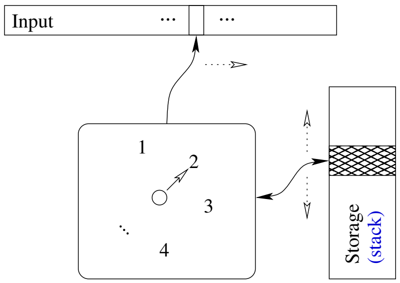


## Pushdown automata

- $M = (K, \Sigma, \Gamma, \Delta, s, F)$
  - $K$, $\Sigma$, $s$, $F$ as before (for finite automata)
  - $\Gamma$ is the stack alphabet
  - $\Delta \subseteq \{(K \times (\Sigma \cup \{\varepsilon\}) \times \Gamma^*) \times (K \times \Gamma^*)\}$
  - Transition:

$$((q, a, \gamma), (q', \gamma'))$$

with $a$ the current input symbol (or $\varepsilon$), $\gamma$ the old stack head, and $\gamma'$ the replacement head




- Configuration: a member of $K \times \Sigma^* \times \Gamma^*$
- $(q, w, u) \vdash_M (q', w', u')$ iff $\exists((q, a, \gamma), (q', \gamma')) \in \Delta : w = aw'$, $u = \gamma x$, $u' = \gamma' x$ for some $x \in \Gamma^*$
- $M$ accepts $w$ iff $(s, w, \varepsilon) \vdash_M^* (f, \varepsilon, \varepsilon)$ for some $f \in F$
- The language accepted by $M$ is

$$\mathcal{L}(M) = \{w \in \Sigma^* : (s, w, \varepsilon) \vdash_M^* (f, \varepsilon, \varepsilon) \text{ for some } f \in F\}$$



## Pushdown automata and CF languages

**Theorem**

*Pushdown automata accept exactly all the context-free languages*

- Construct a finite automaton $M = (K, \Sigma, \Gamma, \Delta, s, F)$ out of a grammar $G = (V, \Sigma, S, R)$ and the other way around
- $\supseteq$
  - $\Gamma = V$, $K = \{p, q\}$, $s = p$, $F = \{q\}$
  - $\Delta = \{((p, \varepsilon, \varepsilon), (q, S))\} \cup \{((q, \varepsilon, A), (q, \alpha)) : A \to \alpha \in R\}$ $\cup \quad \{((q, a, a), (q, \varepsilon)) : a \in \Sigma\}$
  - Complete proof on page 138
- $\subseteq$
  - We work with simplified automata: $((q, a, \gamma), (q', \gamma')) \in \Delta \Rightarrow \gamma \in \Gamma \land |\gamma'| \leq 2$ for any $q \neq s$
  - Given a normal automaton it is easy to construct the simplified automaton $M' = (K', \Sigma, \Gamma \cup \{Z\}, \Delta', s', \{f'\})$, with $K' = K \uplus \{s', f'\}$, $Z \notin \Gamma$ and $\Delta'$ contains for a starter the transitions $((s', \varepsilon, \varepsilon), (s, Z))$ and $((f, \varepsilon, Z), (f', \varepsilon))$ for any $f \in F$



## PDA and CF languages (cont'd)

- $\subseteq$ (cont'd)
  - We add then to $\Delta'$ all the transitions in $\Delta$ that are already in the desired form
  - For any $((q, a, \gamma), (q', \gamma')) \in \Delta$ such that $\gamma = \gamma_1\gamma_2 \ldots \gamma_n$ for some $n > 1$ we add in $\Delta'$:

$$((q, \varepsilon, \gamma_1), (q_1, \varepsilon)) \quad ((q_1, \varepsilon, \gamma_2), (q_2, \varepsilon)) \quad \ldots$$
$$((q_{n-2}, \varepsilon, \gamma_{n-1}), (q_{n-1}, \varepsilon)) \quad ((q_{n-1}, a, \gamma_n), (q', \gamma'))$$

  - For any $((q, a, \gamma), (q', \gamma')) \in \Delta \cup \Delta'$ such that $\gamma' = \gamma_1\gamma_2 \ldots \gamma_m$ for some $m > 2$ we add/replace in $\Delta'$:

$$((q, a, \gamma), (q_1, \gamma_m)) \quad ((q_1, \varepsilon, \varepsilon), (q_2, \gamma_{m-1})) \quad \ldots$$
$$((q_{m-2}, \varepsilon, \varepsilon), (q_{m-1}, \gamma_2)) \quad ((q_{n-1}, \varepsilon, \varepsilon), (q', \gamma_1))$$

  - For any $((q, a, \varepsilon), (q', \gamma)) \in \Delta \cup \Delta'$ we add/replace in $\Delta'$:

$$((q, a, A), (q', \gamma A)) \text{ for all } A \in \Gamma \cup \{Z\}$$

# PDA and CF languages (cont'd)

- $\subseteq$ (cont'd)
  - Now we take $V = \{S\} \uplus \{\langle q, A, p\rangle : q, p \in K', A \in \Gamma \cup \{\varepsilon, Z\}\}$
  - Every nonterminal $\langle q, A, p\rangle$ corresponds to the input consumed by the automaton starting from state $q$ with $A$ at the top of the stack and ending in state $p$
  - Then $R$ is constructed as follows:
    - $S \to \langle s, Z, f'\rangle$
    - For each $((q, a, B), (r, C)) \in \Delta$, $B, C \in \Gamma$ we add $\langle q, B, p\rangle \to a\langle r, C, p\rangle$ for each $p \in K'$
    - For each $((q, a, B), (r, CC')) \in \Delta$, $B, C, C' \in \Gamma$ we add $\langle q, B, p'\rangle \to a\langle r, C, p\rangle\langle p, C', p'\rangle$ for each combination $p, p' \in K'$
    - For each $q \in K'$ we add $\langle q, \varepsilon, q\rangle \to \varepsilon$



# Closure properties

- Consider two grammars with axioms $S_1$ and $S_2$; construct a grammar with axiom $S$
- Context-free languages are closed under
  - Union: Add rules $S \to S_1$ and $S \to S_2$
  - Concatenation: Add rule $S \to S_1 S_2$
  - KLeene star: Add rules $S \to \varepsilon$ and $S \to SS_1$
- Context-free languages are closed under intersection with regular languages
  - $M_1 = (K_1, \Sigma, \Gamma_1, \Delta_1, s_1, F_1) \quad \mathcal{L}(M_1) = L_1$
  - $M_2 = (K_2, \Sigma, \delta_2, s_2, F_2) \quad \mathcal{L}(M)_2 = L_2$
  - Construct $M = (K, \Sigma, \Gamma, \Delta, s, F) \quad \mathcal{L}(M) = L_1 \cap L_2$
  - $K = K_1 \times K_2$, $\Gamma = \Gamma_1$, $s = (s_1, s_2)$, $F = F_1 \times F_2$

$$((q_1, a, \gamma), (p, \gamma')) \in \Delta_1 \quad \Rightarrow \quad (((q_1, q_2), a, \gamma), ((p, \delta_2(q_2, a)), \gamma')) \in \Delta$$
$$((q_1, \varepsilon, \gamma), (p, \gamma')) \in \Delta_1 \quad \Rightarrow \quad (((q_1, q_2), \varepsilon, \gamma), ((p, q_2), \gamma')) \in \Delta$$

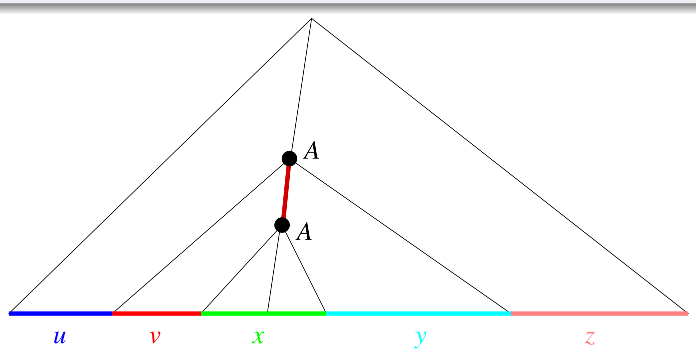


# Pumping context-free languages

- Let $\Phi(G)$ be the maximum fanout (branching factor) of any node in any parse tree constructed based on grammar $G$
- A parse tree of height $h$ has a yield of size no more than $\Phi(G)^h$

**Theorem (pumping context-free languages)**

*For any $w \in \mathcal{L}(G)$ such that $|w| \geq \Phi(G)^{|V-\Sigma|}$ we can write $w$ as $uvxyz$ such that $vy \neq \varepsilon$ and $uv^nxy^nz \in \mathcal{L}(G)$ for any $n \geq 0$*






# Pumping context-free languages (cont'd)

- Some interesting non-context-free languages:
  $\{a^nb^nc^n : n \geq 0\}$
  $\{w \in \{a, b, c\}^* : |w|_a = |w|_b = |w|_c\}$
  $\{a^n : n \text{ is prime}\}$

**Corollary**

*Context-free languages are not closed under intersection and complementation*

- Indeed, $\{a^nb^nc^n : n \geq 0\} = \{a^nb^nc^m : n, m \geq 0\} \cap \{a^mb^nc^n : n, m \geq 0\}$
- That $\{a^nb^nc^m : n, m \geq 0\}$ is context free can be shown by constructing a grammar/automaton or by using closure properties

- Then $\{w \in \{a, b, c\}^* : |w|_a = |w|_b = |w|_c\}$ can be shown non-context-free using closure properties