Proof of NP-completeness of Dominating Set problem

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1. Introduction

Given a graph G(V,E) and a subset V_1 of all vertices, if each vertex in G is either in V_1, or is adjacent to a vertex in V_1, then V_1 is a Dominating Set of graph G(V,E). The Dominating Set Problem is, does a given graph G(V,E) have a dominating set of a given size k ? To prove NP-completeness of this problem, we need to prove it is NP and NP-hard. [1]

2. Proof of NP

Given a certificate of an instance of dominating set problem and the certificate includes the subset V_1, we need to check every vertex in the graph is either in V_1, or adjacent to a vertex in V_1. It is obvious that these checks can be done within a polynomial time, and therefore it is a NP problem. [1]

3. Proof of NP-hard

To prove dominating set problem is NP-hard, we need to show that a known NP-Complete problem is reducible to dominating set problem. In this proof we choose Vertex Cover, which is a known NP-Complete problem.

A vertex cover in an undirected graph G(V,E) is a subset of vertices V_1 such that each edge in graph G has at lease one vertex in V_1. The vertex cover problem is, does a given undirected graph G(V,E) have a vertex cover of a given size k ? [1]

The two problems look quite similar. But there is a key difference if the graph has isolated vertices. In vertex cover problem, these isolated vertices can be ignored since they do not belong to any edges and therefore do not need to be covered. In dominating set problem, however, those vertices should be included in the dominating set, since no vertices would be adjacent to them. [1]

To reduce from vertex cover problem to dominating set problem, we need to construct a new graph G_1 from a given graph G. At the beginning, the new graph is a duplicate of the old one. Then we add an extra vertex w for each edge (u,v), and two edges (u,w), (w,v) to form a triangle to replace the original edge (u,v). This process is illustrated on below Figure 1. [1]

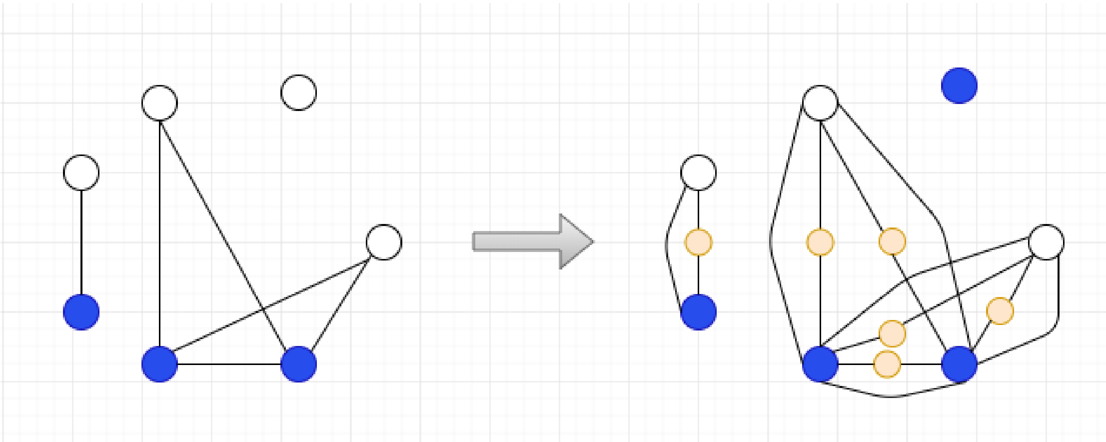

Figure 1. Create new graph for DS from a graph for VC

With this we can show that graph $G_1(V_1,E_1)$ has a vertex cover V_{11} of size k_1 if and only if graph $G_2(V_2,E_2)$ has a dominating set $V_{21}= V_{11} \cup V_i$ of size $k_2= k_1+k_i$, while i means isolated vertices.

First of all, if V_{11} is a vertex cover for G_1, then V_{21} is a dominating set for G_2. There are three kinds of vertices in V_{21}: isolated vertices, original non-isolated vertices , and additionally added vertices on each original edge. All isolated vertices are dominated since they are all in V_{21}. All original non-isolated vertices are also dominated because they are covered by V_{11}, i.e. either in $V_{21}(V_{11})$ or adjacent to a vertex in $V_{21}(V_{11})$ through an edge. As for the additionally added vertices, they are adjacent to both ends of the edges they were added on, and at least one of the both end vertices are in V_{11}, so each additionally added vertex is adjacent to at least one vertex in V_{11}, and therefore dominated. [1]

Secondly, if G_2 has a dominating set V_{21} of size $k_2= k_1+k_i$, then graph G_1 has a vertex cover V_{11} of size k_1. If $V_{22}= V_{21}- V_i$ only contains non-isolated original vertices of graph G_1, then it is a vertex cover. If V_{22} contains additionally added vertices , we can replace them with one of the vertices on the edges which it was added on. Let's say the additional vertices are w, and the both ends of the edge they are added on are (u,v). The vertex w only dominates vertices u,v,w, and the three are still dominated if we replace w with u or v. After replacing, the set V_{22} is obvious vertex cover. This flow is illustrated on below figure 2. [1]

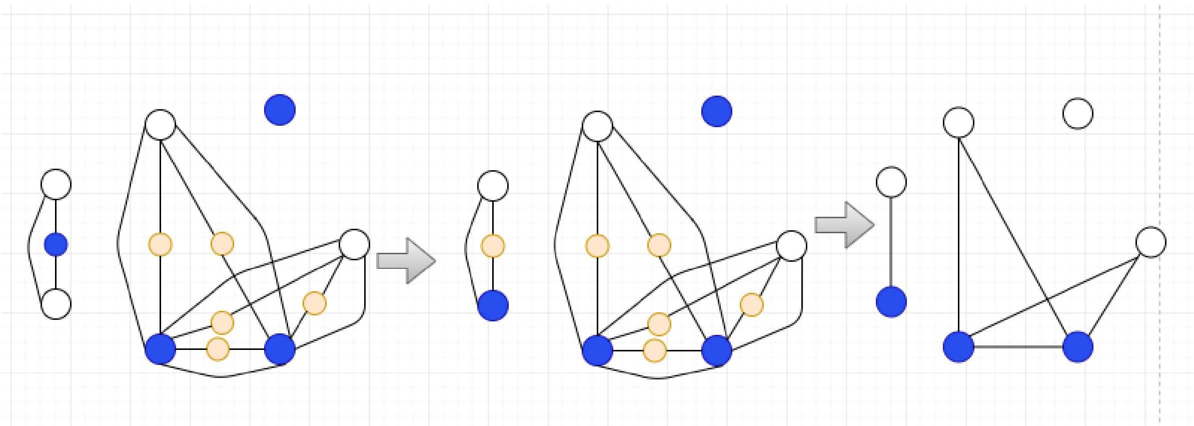

Figure 2. Replace additional vertex with original non-isolated one

4. Optimization problem

The optimization problem related to the decision problem of dominating set is the Minimum Dominating Set (MDS) problem. The Minimum Dominating Set problem is to find a dominating set S of minimum size for a given graph G, i.e. There does not exist a vertex u∊S such that S-u is still a dominating set for graph G. The Minimum Dominating Set problem is NP-hard [2]

References

1. CMSC 451: Lecture 21, NP-Completeness: Clique, Vertex Cover, and Dominating Set http://www.cs.umd.edu/class/fall2017/cmsc451-0101/Lects/lect21-np-clique-vc-ds.pdf
2. Fabian Kuhn, and Roger Wattenhofer. Constant-Time Distributed Dominating Set Approximation