Bishop's University

Math 209: Final Exam
Winter 2015

Name: ______________________________

Student #: ______________________________

- Prepare neat solutions. Briefly justify your work, that is, *make your reasoning clear.*
- All answers must be exact (no decimals allowed) unless specifically directed otherwise.
- This test is 180 minutes in length.
- Do not remove any pages from this test.
- The back of each page may be used for scrap paper.

+ / ×	0	1	2	3	4	5	6	7	8	9	10
0	0 / 0	1	2	3	4	5	6	7	8	9	10
1	0	2 / 1	3	4	5	6	7	8	9	10	11
2	0	2	4 / 4	5	6	7	8	9	10	11	12
3	0	3	6	6 / 9	7	8	9	10	11	12	13
4	0	4	8	12	8 / 16	9	10	11	12	13	14
5	0	5	10	15	20	10 / 25	11	12	13	14	15
6	0	6	12	18	24	30	12 / 36	13	14	15	16
7	0	7	14	21	28	35	42	14 / 49	15	16	17
8	0	8	16	24	32	40	48	56	16 / 64	17	18
9	0	9	18	27	36	45	54	63	72	18 / 81	19
10	0	10	20	30	40	50	60	70	80	90	20 / 100

Page	Points	Score
2	11	
3	12	
4	11	
5	9	
6	9	
7	10	
8	7	
Total:	69	

1. Consider the following matrix A and its reduced form U

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 4 & 5 & 6 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & 0 & -5 & -7 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (7 points) Find a basis for col A, null A, and col A^T.

(b) (4 points) Find an orthogonal basis for col A.

2. (5 points) Let S be the set of all continuous function on $\mathbb{R}$ which are periodic with period p, that is $S = \{f \in C(\mathbb{R}) | f(x+p) = f(x)\}$. Verify that S is a subspace of $C(\mathbb{R})$.

3. Consider the following matrix

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

and let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be two bases for row B.

(a) (4 points) Find the change-of-coordinates matrix from S to T.

(b) (3 points) Let $[\vec{x}]_S = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Find $[\vec{x}]_T$ and $\vec{x}$.

4. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix}$.

(a) (8 points) Find the eigenvalues and associated eigenvectors of A, and state the algebraic multiplicity of each eigenvalue.

(b) (3 points) Find the dimension of each eigenspace of A. Is the matrix diagonalizable?

5. Let $M = \begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix}$.

(a) (6 points) Find an invertible matrix P and a diagonal matrix D such that $M = PDP^{-1}$.

(b) (3 points) Write the spectral decomposition of M.

6. (3 points) Let the matrix A be $m \times n$ and B be an $n \times p$ matrix. Show that the column space of AB is a subspace of the column space of A.

7. (3 points) Why is $\langle \vec{u}, \vec{v} \rangle = u_1 v_2 + u_2 v_1$ not an inner product on $\mathbb{R}^2$?

8. (3 points) Prove that if A is an $m \times n$ matrix and the linear transformation $T(\vec{x}) = A\vec{x}$ is onto, then rank $A = m$.

9. Consider the function $\langle \vec{u}, \vec{v} \rangle = u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2$.

(a) (5 points) Verify that $\langle \vec{u}, \vec{v} \rangle$ is an inner product on $\mathbb{R}^2$

(b) (3 points) Find a vector $\vec{x}$ which is orthogonal to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ with respect to this inner product.

(c) (2 points) Find an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product.

10. Consider the matrix equation $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \qquad \text{and} \qquad \vec{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}.$$

(a) (5 points) Find the least-squares solution to $A\vec{x} = \vec{b}$.

(b) (2 points) Find the distance from $\vec{b}$ to col A.