Bishop's University

Math 209: Final Exam
Winter 2022

Name: ______________________________

Student #: ______________________________

- Prepare neat solutions. Briefly justify your work, that is, *make your reasoning clear.*
- All answers must be exact (no decimals allowed) unless specifically directed otherwise.
- This exam is 180 minutes in length.
- Do not remove any pages from this test.
- The back of each page may be used for scrap paper.
- A **Casio fx260-solar** or **Casio fx260-solar II** calculator is permitted. No other aids are permitted.
- **Remember that Bishop's University has a ZERO-TOLERANCE POLICY for academic misconduct on final exams.**

Page	Points	Score
2	10	
3	15	
4	10	
5	10	
6	10	
7	10	
8	10	
9	15	
Total:	90	

1. (5 points) Let V be the vector space of all 2×2 real matrices, and let H be the set of all 2×2 real matrices with trace equal to zero. Is H a subspace of V? Justify your answer.

2. (5 points) Are the vectors $5 + 4x - 3x^2$, $1 + 5x - 4x^2$, and $-4 + x + 7x^2$ linearly independent? If not, find a basis for the span of these vectors.

3. Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 4 & 1 \\ -4 & 2 & -8 & -2 \\ 3 & 3 & -9 & 0 \\ -3 & 3 & -11 & -2 \end{bmatrix}$$

with reduced echelon form

$$R = \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{10}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (5 points) Find a basis for the column space of A.

(b) (5 points) Find a basis for the row space of A.

(c) (5 points) Find an **orthonormal basis** for the null space of A.

4. (10 points) Consider the matrices

$$A = -\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Find an orthogonal matrix V such that $A = V\Lambda V^t$.

5. (10 points) Let V be the vector space of 2×2 real matrices, and define $\langle A, B \rangle = \text{tr}\ (A^t B)$. Show that this is an inner product.

6. (10 points) Find the orthogonal projection of $\vec{v} = (0, 0, 0, -9)$ onto the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 1, 1), (1, -1, -1, 1)$, and $(1, 0, -1, 0)$.

7. (10 points) Let $P_3(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 3. Let $B = \{9, 6x, -4x^2, 5x^3\}$ and $C = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ be ordered bases for $P_3(\mathbb{R})$. Find the transition matrix $P_{C\leftarrow B}$.

8. (5 points) The set

$$\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

is a basis for the vector space of upper-triangular real 2×2 matrices. Find the coordinate vector for

$$A = \begin{bmatrix} 7 & -9 \\ 0 & 7 \end{bmatrix}$$

with respect to this basis.

9. (5 points) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(3, 4) = (13, 21)$ and $T(2, -5) = (24, -9)$, find the standard matrix of T.

10. (5 points) Find the matrix A of the linear transformation $T(f(t)) = f(9t+3)$ from the vector space V to V with respect to the basis $\{1, t, t^2\}$ for V.

11. (10 points) Find the least squares solution $\hat{x}$ of the system

$$\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ -4 \\ 20 \end{bmatrix}.$$