Bishop's University

Math 209: Final Exam
Winter 2024

Name: ______________________________

Student #: ______________________________

- This exam is 180 minutes in length.
- All solutions must be written on this exam paper. No extra paper is permitted.
- All answers must be exact (no decimals allowed) unless specifically directed otherwise.
- Prepare neat solutions. Briefly justify your work, that is, *make your reasoning clear.*
- You are permitted to use one (1) **Authorized Memory Book** and a **Casio fx-260 Solar (II) calculator**.
- Do not remove any pages from this test.
- All answers must be written in the space provided.
- The back of each page may be used for scrap paper.
- **Remember that Bishop's University has a ZERO-TOLERANCE POLICY for academic misconduct on final exams.**

Page	Points	Score
2	10	
3	15	
4	10	
5	15	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (5 points) Let the matrix A be $m \times n$ and B be an $n \times p$ matrix. Show that rank $AB \leq$ rank A. (Hint: Show that the column space of AB is a subset of the column space of A.)

2. (5 points) Consider the vector $\vec{v} = (1, 3, -1)$ in $\mathbb{R}^3$ and let W be the subspace of $\mathbb{R}^3$ consisting of all vectors of the form $(a, b, a - 2b)$. Decompose $\vec{v}$ into the sum of a vector that lies in W and a vector orthogonal to W.

3. Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 4 & 1 & 3 \\ -4 & 2 & -8 & -2 & 5 \\ -6 & 3 & -9 & 0 & 4 \end{bmatrix}$$

with reduced echelon form

$$R = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) (5 points) Find a basis for the column space of A.

(b) (5 points) Find a basis for the row space of A. The basis vectors must have integer components.

(c) (5 points) Find an **orthonormal basis** for the null space of A.

4. (10 points) Consider the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 2 & -2 & 2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Find an orthogonal matrix V such that $A = V\Lambda V^t$.

5. Let V be the vector space of 2×2 real matrices and define $\langle A, B \rangle = \operatorname{tr}(A^t B)$.

   (a) (10 points) Show that this is an inner product.

   (b) (5 points) With reference to this inner product, find the distance between $\begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$ and $\begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}$

6. (10 points) Find the orthogonal projection of $\vec{v} = (1, -2, 3, -4)$ onto the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 1, 1), (1, 1, -1, -1)$, and $(1, 0, 1, 0)$.

7. (10 points) Let $P_3(\mathbb{R})$ be the vector space of all polynomials of degree less than or equal to 3. Let $B = \{6, -3x, -4x^2, 2x^3\}$ and $C = \{1, 1-x, 1-x+x^2, 1-x+x^2-x^3\}$ be ordered bases for $P_3(\mathbb{R})$. Find the transition matrix $P_{B \leftarrow C}$.

8. (5 points) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(3, 2) = (13, 21)$ and $T(2, 3) = (24, -9)$, find the standard matrix of T.

9. (5 points) Find the matrix A of the linear transformation $T(f(t)) = f(1 - t)$ from the vector space V to V with respect to the basis $\{t, t - 1, t^2 - t - 1\}$ for V.

10. (10 points) Find the least squares line for the data set $\{(1, -2), (3, 1), (0, 3), (-2, 5)\}$, and use this to estimate the y-coordinate associated with $x = 2$.