Math 315 : Real Analysis
Final Exam (April 23, 2019)

1. Define
   (a) (5 points) a neighbourhood of a real number x.
   (b) (5 points) the limit of a function f at an accumulation point x_0.
   (c) (5 points) uniformly continuous.
   (d) (5 points) an infinite series.
   (e) (5 points) the Cauchy Product of two infinite series.

2. State clearly and concisely each of the following:
   (a) (5 points) the Bolzano-Weierstrass Theorem
   (b) (5 points) the Heine-Borel Theorm
   (c) (5 points) the Intermediate Value Theorem

3. Give an example of each of the following. **No justification is required.**
   (a) (2 points) a closed set which is not compact.
   (b) (2 points) a bounded set which is not compact.
   (c) (2 points) a continuous function on a bounded interval which is not uniformly continuous.
   (d) (2 points) a conditionally convergent infinite series.
   (e) (2 points) a non-empty open set which contains all its accumulation points.

4. (5 points) Prove that between any two real numbers there exists an irrational number.

5. (5 points) Show that the sequence defined by $a_1 = 6$ and $a_n = \sqrt{6 + a_{n-1}}$ for $n > 1$ is convergent and find the limit.

6. (5 points) Suppose $f : D \to \mathbb{R}$ with x_0 and an accumulation point of D. Assume that L_1 and L_2 are limits of f at x_0. Use the definition of limit to prove $L_1 = L_2$.

7. (5 points) Let $E \subseteq \mathbb{R}$. Prove E is closed if it contains all limits of sequences of members of E.

8. (5 points) Determine the values of r for which $\displaystyle\sum_{n=0}^{\infty} nr^n$ converges.

9. (5 points) Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+64n}+\sqrt{n^2+3}}$ converges. (**Hint: the limit-comparison test may be helpful.**)

10. (5 points) Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} 3^{-n}\sqrt{n}x^n$.