Bishop's University

Math 317: Final Exam
Fall 2022

- This exam is 180 minutes in length.
- Prepare neat solutions. Briefly justify your work, that is, *make your reasoning clear.*
- All answers must be written in the space provided.
- The back of each page may be used for scrap paper.
- All answers must be exact (no decimals allowed) unless specifically directed otherwise.
- All answers must be in the form of $x + iy$ unless otherwise indicated.
- You are permitted to use one (1) Authorized Memory Book and a Casio fx-260 Solar (II) calculator.
- **Remember that Bishop's University has a ZERO-TOLERANCE POLICY for academic misconduct on final exams.**

Page	Points	Score
2	10	
3	10	
4	25	
5	10	
6	10	
7	10	
8	10	
9	15	
Total:	100	

1. (5 points) Write $(-1+i)^{-4}$ in complex exponential form with an argument in the interval $(\pi, 3\pi]$.

2. (5 points) Write $\dfrac{e^{i\frac{1}{3}\pi}}{e^{i\frac{1}{4}\pi}}$ in rectangular form. All trigonometric functions must be evaluated exactly.

3. (5 points) Express in the form $a + bi$, $a, b \in \mathbb{R}$, all solutions of $z^6 - 19z^3 = 216$. All trigonometric functions must be evaluated.

4. (5 points) Solve: $\sinh z = i$.

5. Let $f(z) = \overline{z}$ and $g(z) = \dfrac{1}{|z|^2}$.

   (a) (5 points) Prove that $f(z)$ is not analytic.

   (b) (5 points) Prove that $g(z)$ is not analytic.

   (c) (5 points) Prove that $f(z)g(z)$ is analytic on its domain.

6. (10 points) If $f(z) = u(x, y) + iv(x, y)$ and $g(z) = a(x, y) + ib(x, y)$ are analytic functions, use the Cauchy-Riemann equations to prove that $f(z)g(z)$ is analytic.

7. (10 points) Evaluate $\oint_C \frac{\sin z}{z^2(z^2+9)(z+3)^2}$ where C is $|z| = 1$, travelled clockwise.

8. (5 points) Without actually obtaining the series, determine the radius of convergence of the given Taylor series. What is the disk on which the indicated series converges to the given function?

$$\frac{1}{z^2+z+1} = \sum_{n=0}^{\infty} a_n (z-i)^n$$

9. (5 points) Find all Laurent series for

$$f(z) = \frac{1}{z^2(1-z)}$$

about $z = 0$ and state where they are convergent.

10. (5 points) Evaluate $\oint_C z^8 \sin\left(\frac{1}{z}\right) dz$ where C is $|z - i| = 4$, travelled counterclockwise.

11. (5 points) Compute $\oint_C \frac{e^z}{\sin \pi z} dz$ where C is the curve $|z| = \sqrt{2}$ travelled counterclockwise.

12. (10 points) Evaluate $\int_{-\infty}^{\infty} \frac{x}{(x^2 - 2x + 2)^2} dx$

13. Consider a linear fractional transformation that maps

$$\begin{aligned} -1 &\text{ to } -1 \\ i &\text{ to } -i \\ 1+i &\text{ to } 1. \end{aligned}$$

(a) (8 points) Find the transformation.

(b) (7 points) What are the fixed points?