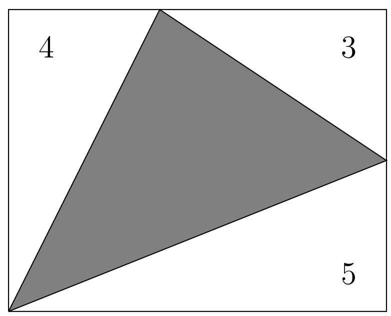
Math 326: Mathematical Problem Solving
Final Exam, Winter 2022

Instructions:

> **Twelve (12) problems were chosen at random. The original numbering remains the same. Complete any eight (8), justifying your answers. A Casio FX-260 solar calculator may be used.**

3. Consider the following diagram of a rectangle:



Find the area of the grey triangle, given that the white triangles have areas as labelled in the diagram.

4. Let X be a set with at least six elements, and let $E_1, E_2, \ldots, E_6$ be six subsets of X each containing exactly three elements. Show that it is possible to colour the elements of X in two colours such that each E_i contains two elements of different colours.

5. Let E be the set consisting of all numbers from 1 to 1000000, inclusive, and let A be the set of all elements of E which can be written as the sum of a square plus a cube. For example, $1009 = 3^2 + 10^3$ is an element of A. Which is bigger, the set A or the complement of A in E?

6. Let P be a point on the graph of $y = f(x)$ where f is a third degree polynomial. Let the tangent of $f(x)$ at P intersect the curve again at Q, and let A be the area of the region bound by the curve and the segment PQ. Let B be the area of the region defined in the same way starting with Q instead of P. What is the relation between A and B?

7. Consider the functional equation:

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right).$$

Find all possible differentiable functions f which makes the statement true.

8. Let A be an $n \times n$ real matrix, and suppose $A^3 = A + I$. Prove $\det(A) > 0$.

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9. Consider the following game: You are given N vertices and allowed to build a graph by adding edges connecting these vertices. For each edge you add, you make \$1. You are not allowed to add loops or multiple edges joining the same pair of vertices, and you must stop before the graph is connected. What is the most money you can make building the graph? Give your answers in terms of N.

10. Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for relatively prime integers m and n, then $m - n$ is divisible by p.

11. Consider the table

$$\begin{aligned} 1 &= 1 \\ 3 + 5 &= 8 \\ 7 + 9 + 11 &= 27 \\ 13 + 15 + 17 + 19 &= 64 \\ 21 + 23 + 25 + 27 + 29 &= 125 \end{aligned}$$

Guess the general law suggested by these examples, express it in suitable mathematical notation, and prove it.

13. Solve

$$y' + e^y = 2x + \frac{1}{x}.$$

14. Without using a calculator to evaluate, determine which is larger: $2^{\sqrt{7}}$ or $3^{\frac{13}{8}}$?

15. A penny is moved in the cartesian plane according to the following constraints:

- The penny starts at the point $(1, 1)$
- From any point (i, j), the penny can move to $(2i, j)$ or $(i, 2j)$
- From any point (i, j), the penny can move to $(i - j, j)$ if $i > j$, or to $(i, j - i)$ if $j > i$.

For which positive integers x, y can the penny be moved to (x, y). The statement should be an "if and only if", hence two implications must be proved.