Bishop's University

Math 401: Final Exam
Winter 2016

- This exam is due no later than 12:00pm (noon) on April 25, 2016.
- Prepare neat solutions. Briefly justify your work, that is, *make your reasoning clear.*
- All questions are equally weighted.

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1. (5.1 #12) Let
$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
Show that the derivative,
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
exists for all x but that $f'(x)$ is not a continuous function. (Hint: the limit definition is only necessary at $x = 0$.)

2. (5.2 # 10) The map
$$\begin{aligned} x &= r\cos\theta\cos\phi \\ y &= r\sin\theta\cos\phi \\ z &= r\sin\phi \end{aligned}$$
defines spherical coordinates on xyz-space.
   (a) Find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.
   (b) At what points is the map non-singular of rank three? What are the ranks of the remaining points?
   (c) Using implicit differentiation, find the partial derivatives of r, θ, and ϕ with respect to x, y, and z.
   (Notes: There should be nine derivatives in total. Use the combinations $\cos\theta\, \mathrm{d}x + \sin\theta\, \mathrm{d}y$ and $-\sin\theta\, \mathrm{d}x + \cos\theta\, \mathrm{d}y$ as mentioned on p. 146 of the text. Assume that $r > 0$ and $\cos\phi > 0$.)

3. (5.4 #10) Is the quadratic form $Q(x,y) = 5x^2 + 6xy + 2y^2$ elliptic, hyperbolic, or parabolic? (Check p. 178 for definitions.) Sketch the curves $Q =$ constant, showing axes and maximum and minimum values of Q on the circle $x^2 + y^2 = 1$

4. (6.2 #1)Given that the area of the unit circle is π, use pullbacks to find the area of the ellipse
$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+2}{7}\right)^2 = 1.$$

5. (6.2 #2) Use pullbacks to find the volume of the tetrahedron with vertices $(-1,-2,1)$, $(-2,1,3)$, $(1,1,1)$, and $(0,0,-4)$.

6. (8.1 #1c) Find the vector field grad$f$ for $f(x,y,z) = \ln(x^2 + y^2)$ and write as a 2-form in xyz-space.

7. (8.1 #3) Let $\vec{X} = (x^2+1)\vec{\imath} + (xyz)\vec{\jmath} + \sin(x+y)\vec{k}$. Write $\vec{X}$ as a 1-form and as a 2-form in xyz-space, and use these to compute div$\vec{X}$ and curl$X$.

8. (8.2 #1b) Find the function $F(x,y)$ such that the curves $F =$ constant solves the differential equation
$$x\mathrm{d}x + \frac{1+x^2}{1+y^2}\mathrm{d}y = 0.$$

9. (8.2 #2d) A differential equation gives a concise description of a family of curves. For example, $x\mathrm{d}x + y\mathrm{d}y = 0$ describes concentric circles about the origin. Find a differential equation (written using differential forms) which describes ellipses with center at $(0,0)$ with the major axis along the x-axis, and with the major axis twice as long as the minor axis.