Bishop's University

Math 406: Final Exam
Winter 2021

- This exam is due no later than 11:59pm on April 30, 2021.
- Your solutions are to be scanned into a single PDF document and uploaded to Moodle.
- Prepare neat solutions. Briefly justify your work, that is, *make your reasoning clear.*
- Marks will be awarded for progress towards a solution as well as for the solution of a problem.
- The work must be your work, Make sure you are aware of the plagiarism policy outlined in the Academic Calendar.

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1. (5 points) Consider a curve $\mathbf{p}(t)$ in the (x, y)-plane. Prove that

$$\kappa(t) = \frac{(\mathbf{p}'(t) \times \mathbf{p}''(t)) \cdot \mathbf{k}}{|\mathbf{p}'(t)|^3}.$$

2. Consider the curve, C, given by $x^4 + y^4 = 2$.
   (a) (5 points) Find the curvature $\kappa(t)$ and evaluate at the point $P(-1, 1)$.
   (b) (5 points) Find the vertices of C.
   (c) (5 points) Find the total curvature μ of C.
   (d) (5 points) Determine the width of the curve C.

3. Let $\mathbf{p}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$.
   (a) (5 points) Find $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$.
   (b) (5 points) Compute $\kappa(t)$ and $\tau(t)$.
   (c) (5 points) Verify the Frenet-Serret's formula for this curve.

4. Consider the surface, S, parameterized by $\mathbf{p}(u, v) = \langle u - 2v, u^2 + v^2, u + 2v \rangle$.
   (a) (5 points) Compute the first and second fundamental forms.
   (b) (5 points) Calculate the Christoffel's symbols.
   (c) (5 points) Compute K and H.
   (d) (5 points) Find the 1-forms θ^1, θ^2, and ${\omega_2}^1$ and verify that $d{\omega_2}^1 = K\theta^1 \wedge \theta^2$.

5. Let $ds^2 = dr \wedge dr + \sinh^2 r d\theta \wedge d\theta$ be a metric on the domain $H = \{(r, \theta) \mid r > 0, 0 \le \theta < 2\pi\}$.
   (a) (5 points) Find the Gaussian curvature.
   (b) (5 points) Determine the parallel vector fields along a curve $r = a$, $\theta = t$ with $a > 0$ and $0 \le t < 2\pi$.
   (c) (5 points) Calculate the geodesics on H with this metric.

6. The differential 1-form

$$\phi = \frac{udv - vdu}{u^2 + v^2}$$

   is defined on the punctured (u, v)-plane (the (u, v)-plane excluding the origin.)
   (a) (5 points) Prove that $d\phi = 0$.
   (b) (5 points) Integrate ϕ along the unit circle, centered at the origin, with positive (counter-clockwise) orientation.
   (c) (5 points) Does there exist a function f such that $df = \phi$ on the punctured (u, v)-plane?