Bishop's University

Math 431: Final Exam
Winter 2017

- This exam is open book, closed notes.
- Please show all work. If a result is in the exercises for Chapters 1-13,15-16 please reference it (unless it is the question you are asked to prove). Otherwise, justify the result.

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1. (5 points) Prove that if A and B are bounded subsets of a metric space and $A \cap B \neq \emptyset$ then $\text{diam}(A \cup B) \leq \text{diam}A + \text{diam}B$.

2. (5 points) Let $f : X \to Y$ be a continuous map of topological spaces. Show that the image by f of a basis in X is not necessarily a basis in Y.

3. (10 points) Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = n^{-x} x^n \cos(nx)$, for $n \in \mathbb{N}$. Does (f_n) converge uniformly? Why or why not?

4. (10 points) Let A be a subset of a topological space X. A point $x \in X$ is a limit point of A if for every open set U containing x, $(U \setminus \{x\}) \cap A \neq \emptyset$. Now let X be a Hausdorff space, A a subset of X, and x a limit point of A. Prove that every open set containing x contains infinitely many elements of A.

5. (10 points) Let X be a compact space. Prove that if A is a subset of X with no limit points, then A is finite.

6. (a) (10 points) Let $\{\mathcal{T}_i\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all $\mathcal{T}_i$, and a unique largest topology which is contained in all of the $\mathcal{T}_i$.

   (b) (10 points) If $X = \{a, b, c\}$, let $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$. Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

7. (a) (5 points) Let A and B be open sets in a topological space X. Does $\overline{A} = \overline{B}$ imply that $A = B$? Give a proof or a counter-example.

   (b) (5 points) Let A and B be closed sets in a topological space X. Does $A^\circ = B^\circ$ imply that $A = B$? Give a proof or a counter-example.

8. (10 points) Let $f : X \to Y$ be a map of metric spaces. Prove that f is continuous if and only if whenever (x_n) is a sequence in X converging to a point $x \in X$ we have $(f(x_n))$ converges to $f(x)$ in Y.