Bishop's University

Math 462: Final Exam
Fall 2017

- This exam is due no later than 12:00pm (noon) on December 15, 2017.
- Prepare neat solutions. Justify your work, that is, *make your reasoning clear.*
- All questions are equally weighted.

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1. Prove that if x, y, and z are vectors in a Hilbert space, then

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{x+y}{2}\right\|^2.$$

2. Let f be a real-valued function on a metric space X. Show that f is continuous if and only if for every α the sets $\{x \in X | f(x) < \alpha\}$ and $\{x \in X | f(x) > \alpha\}$ are open. Let g be a function that is the pointwise limit of a sequence of continuous functions $f_n : X \to \mathbb{R}$. Suppose that for each λ the sequence $\{f_n(\lambda)\}$ is increasing. Show that then the sets $\{x \in X | g(x) > \alpha\}$ are open, but show by means of an example that g need not be continuous.

3. Prove that functions with continuous derivative form a Euclidean space under the norm

$$\|f\| := \left(\int_0^1 x\,|f(x)|^2 + 2\,|f'(x)|^2\,dx\right)^{1/2}.$$

4. Show that the unit ball in a Euclidean space is compact if and only if the space is finite dimensional.

5. Let a, b, c, and x be points in a Hilbert space such that $b = \frac{1}{2}(a+c)$ and

$$0 < r \le \|x-a\| \le \|x-b\| \le \|x-c\| \le r+\epsilon \le 2r.$$

   Show that $\|a-c\|$ goes to zero as ϵ goes to zero. **(Hint: use the parallelogram law).**

6. Show that for all $x_0 \in X$ there exists an $f \in X^*$ with $\|f\| = \|x_0\|$ and $f(x_0) = \|x_0\|^2$. Show by example that this f is not unique.

7. Give an example of a discontinuous (unbounded) linear functional.

8. Let F be a subspace of a Banach space E. Show that if F is *not* dense, there exists $f \in E^*$ with F contained in the kernel of f.