University of New Brunswick, Saint John
Department of Mathematical Sciences

Fall 2010

Math 1503
Introduction to Linear Algebra
Final Exam: Worth 60% of your final mark

Name: ______________________________

Student #: ______________________________

Time: 3 hours

Mark: / 75

Instructions:

- Do not remove any sheets from this booklet.
- Show all your work in this booklet. For full marks all work must be shown.
- Work neatly and in an organized manner.
- If you run out of space in a problem, use the space on the back of the page and **clearly** indicate where the solution continues.
- Calculators are **NOT** allowed!
- Good luck!

1. Given three points $P(1,1,0), Q(0,1,-1)$, and $R(1,1,1)$, answer the following questions:

(2) (a) Find the vectors $\vec{u} = \vec{PQ}$ and $\vec{v} = \vec{PR}$.

(3) (b) Find the projection of $\vec{u}$ onto $\vec{v}$.

(3) (c) Find a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

(2) (d) Find an equation of the plane containing P, Q, and R, in general (standard) form.

2. Consider the following sytem of linear equations

$$\begin{array}{rcrcrcl} x & - & 2y & + & z & = & 2 \\ & & 2y & - & z & = & 4 \\ x & & & + & 3z & = & 3 \end{array}$$

(1) (a) Write the matrix equation representing the above system of equations.

(6) (b) Find the inverse of the coefficient matrix.

(2) (c) Solve the system using the inverse found above.

(1) (d) Verify your solution using any method discussed in class (except for the inverse method.)

3. Let $A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$.

(1) (a) Find the determinant of A.

(1) (b) Find the trace of A.

(3) (c) Find the eigenvalues of A.

(1) (d) Verify that $\det A$ is the product of the eigenvalues and $\operatorname{tr} A$ is the sum of the eigenvalues.

(2) (e) Find the eigenvectors of A.

(3) (f) Diagonalize A.

(4) (g) Use the diagonalization above to compute A^{-5}.

(5) 4. The matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ has eigenvalues $1 - i$ and $1 + i$. Use your knowledge of eigenvalues and complex numbers to determine the eigenvalues of A^8. Do not compute A^8.

5. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 10 & 0 \\ 10 & 0 & 10 \\ 5 & 0 & 15 \end{bmatrix}$$

(3) (a) Calculate $\det(A)$.

(3) (b) Use the answer above and your knowledge of determinants to compute $\det(B)$. Note: do not compute $\det(B)$ directly.

(4) 6. Determine whether the following set of vectors are linearly dependent or independent. Justify your answer.

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}$$

7. Determine whether each of the following sets a subspace of $\mathbb{R}^3$? Justify your answer.

(5) (a) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, xyz = 0 \right\}$

(5) (b) $T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + 3z = 0 \right\}$

8. Let

$$A = \begin{bmatrix} 1 & 2 & 4 & 2 & 8 \\ 2 & -2 & 2 & -2 & 4 \\ 2 & -2 & 2 & -1 & 2 \\ -2 & 3 & -1 & 2 & 0 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $R = \text{RREF}(A)$.

(2) (a) Find the rank and nullity of A

(4) (b) Find bases for row(A), col(A), and null(A).

(2) (c) Is $\vec{u} = \begin{bmatrix} 6 \\ 5 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ in the null space of A? If so, write $\vec{u}$ as a linear combination of the basis vectors for the null space.

(2) (d) Is $\vec{v} = [1\ 1\ 3\ 2\ 4]$ in the row space of A? If so, write $\vec{v}$ as a linear combination of the basis vectors for the row space.

(5) 9. Do **ONE** of the following questions:

(a) For vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, prove that

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\ \|\vec{u}\|^2 + 2\ \|\vec{v}\|^2 .$$

(b) Let A be a symmetric matrix. Prove that any eigenvalue of A is a real number. (Hint: consider $\|A\vec{x}\|$, where $\vec{x}$ is an eigenvector of eigenvalue λ.)