Math 2013, Winter 2009

FINAL EXAM **APRIL 17, 2009**

STUDENT'S NAME:_____ **ID #:**_____

Note: You must show your work in order to receive full marks.
No electronic devices are allowed.

1. **Vector Calculus**

(10) (a) Show that $\overrightarrow{F}(x,y) = \langle e^x \sin y, e^x \cos y + \sin y\rangle$ is conservative and find a function ϕ such that $\overrightarrow{F} = \nabla\phi$. Use ϕ to evaluate $\int_C \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r}$ where C is the arc of an ellipse going from $(0,0)$ to $(-1, \frac{\pi}{4})$.

(10) (b) Use Green's Theorem to find $\oint_C (y^3\,\mathrm{d}x - x^3\mathrm{d}y)$ where C is the circle $x^2+y^2=4$ travelled counterclockwise.

(c) For this question, let $\overrightarrow{F} = \langle x^2yz, xy^2z, xyz^2\rangle$.

(8) i. Find the divergence and curl of $\overrightarrow{F}$.

(2) ii. Is it possible to express $\overrightarrow{F}$ of part (i) as the gradient of a function f?

(d) For this question, **set-up, but do not evaluate** the integrals.

(5) i. Use Stokes' Theorem to write $\oint_C \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r}$ as a double integral, where $\overrightarrow{F} = \langle 3, z^2, yz\rangle = 3\overrightarrow{i} + z^2\overrightarrow{j} + yz\overrightarrow{k}$ and C is the boundary of the paraboloid $y = 4 - x^2 - z^2$ in the first octant travelled clockwise as viewed from the origin.

(5) ii. Use the Divergence Theorem to write the flux of $\overrightarrow{F} = 2x^3z\overrightarrow{i} + 2y^3z\overrightarrow{j} + 3z^2\overrightarrow{k}$ across the sphere $x^2+y^2+z^2=4$, oriented outward, as a triple integral.

2. **Differential Equations**

(a) Find the general solution of the following differential equations using the specified method:

(10) i. $y'' - y' = e^x \cos x$ (variation of parameters)

(10) ii. $y'' - 3y' + 2y = e^{2x}$ (undetermined coefficients)

3. **Sequences and Series**

(a) Test the following series for convergence or divergence:

(5) i. $\displaystyle\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$

(5) ii. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^{2n+1}}$

(5) iii. $\displaystyle\sum_{i=1}^{\infty} \frac{e^{\frac{1}{i}}}{i^2+i}$

(5) (b) Determine, with justification, whether the following series is absolutely convergent, conditionally convergent or divergent: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3n-2}$

(c) Let $f(x) = \displaystyle\sum_{n=0}^{\infty} (n+1)\frac{x^n}{3^n}$.

(5) i. Find the radius of convergence for f

(10) ii. Find a series for $\int f(x)\,\mathrm{d}x$ and a formula for its sum. **(Hint: geometric series)**

(5) iii. Find a formula for $f(x)$. **(Hint: $f(x)$ is the derivative of $\int f(x)\,\mathrm{d}x$)**