Math 3073 Final Exam, April 20, 2009

1. Consider the following PDE: $u_{ss} - 6u_{sr} = 0$.
   (1) (a) Classify as elliptic, parabolic, or hyperbolic.
   (4) (b) Complete the square to write in standard form. Use x and t as the new variables.
   (6) (c) With respect to the new variables, consider the following initial conditions: $u(x,0) = \sin x$ and $u_t(x,0) = 0$, for $x \in \mathbb{R}$. Solve for u in terms of x and t.
   (1) (d) Use the above solution to solve for u in terms of r and s.

(10) 2. Solve:

$$\begin{aligned} u_t &= 2u_{xx}, && 0 < x < \pi, \quad 0 < t < \infty \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= e^x && 0 < x < \pi \end{aligned}$$

(10) 3. Solve:

$$\begin{aligned} u_{tt} &= 9u_{xx}, && 0 < x < \pi, \quad 0 < t < \infty \\ u(0,t) &= u(x,0) = u_x(\pi,t) = 0 \\ u_t(x,0) &= \cos 2x && 0 < x < \pi \end{aligned}$$

(10) 4. Solve $u_{xx} + u_{yy} = 0$ on the wedge $x^2 + y^2 < a^2$, $x > 0$, $y > 0$ with the following boundary conditions: $u = 0$ on $x = 0$ and $y = 0$, and $u_r = 1$ on $r = a$.

(10) 5. Show directly that the eigenvalues of the operator $\frac{d^2}{dx^2}$ with boundary conditions $X(-\pi) + X'(-\pi) = X(0) + X'(0) = 0$ are real and negative.

(8) 6. Prove the comparison principle for the diffusion equation, $u_t = ku_{xx}$: If u and v are two solutions, and if $u \le v$ for $t = 0$, $x = 0$ and $x = l$, then $u \le v$ for $t \ge 0$ and $0 \le x \le l$.

Helpful formulae:

$$\int_0^\pi e^x \sin ax dx = \frac{a + e^\pi(\sin a\pi - a\cos a\pi)}{1+a^2} \qquad \int_0^\pi \sin^2 ax dx = \frac{a\pi - \cos a\pi \sin a\pi}{2a}$$

$$\int_0^\pi e^x \cos ax dx = \frac{-1 + e^\pi(a\sin a\pi + \cos a\pi)}{1+a^2} \qquad \int_0^\pi \cos^2 ax dx = \frac{a\pi + \cos a\pi \sin a\pi}{2a}$$

$$\int_0^\pi \cos 2x \sin ax dx = \frac{a(\cos a\pi - 1)}{4-a^2} \qquad \int_0^\pi \cos 2x \cos ax dx = \frac{a\sin a\pi}{a^2-4}$$