Math 3073: Final Exam

April 9, 2011

(10%) 1. Solve the following PDE:

$$x^2u_x + xyu_y + xu = x - y.$$

(10%) 2. Find the regions in the xy-plane where the equation

$$(x+1)u_{xx} + 2yu_{xy} - (1-x)u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch the regions.

(20%) 3. Use the method of factoring to solve

$$\left(\frac{\partial^3}{\partial x^3} - 3\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t} - 4\frac{\partial}{\partial x}\frac{\partial^2}{\partial t^2} + 12\frac{\partial^3}{\partial t^3}\right)u(x,t) = 0$$
$$u(x,0) = 0, \quad u_t(x,0) = x, \quad u_{tt}(x,0) = x^2$$

(Hint: $\dfrac{\partial}{\partial x} - 3\dfrac{\partial}{\partial t}$ is a factor of the linear operator. Also, when solving for the arbitrary functions, integrate over the interval $[0, x]$.)

(20%) 4. Solve:

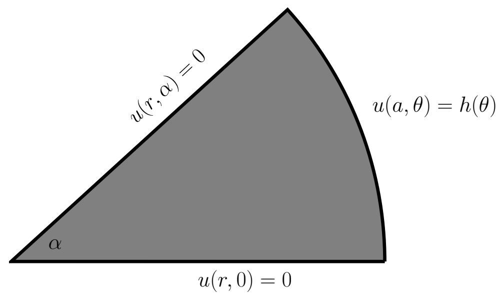
$$\begin{aligned} u_t &= 2u_{xx}, && 0 < x < \pi, \quad 0 < t < \infty \\ u_x(0,t) &= u_x(\pi,t) = 0 && \\ u(x,0) &= \sin x && 0 < x < \pi \end{aligned}$$

(20%) 5. Find all eigenvalues and associated eigenfunctions of $-\dfrac{d^2}{dx^2}$ with the following boundary conditions: $X'(0) + X(0) = X'(1) + X(1) = 0$.

(20%) 6. Consider the following PDE

$$\begin{aligned} \Delta_2 u(r,\theta) &= 0 && \text{for } 0 < r < a, \;\; 0 < \theta < \alpha \\ u(r,0) = u(r,\alpha) &= 0 \\ u(a,\theta) &= h(\theta) \end{aligned}$$

as illustrated in the following diagram. Note that we are using polar coordinates.



(a) Assume that $u(r,\theta) = R(r)T(\theta)$. Use separation of variables to generate ordinary differential equations for R and T. Make sure the negative sign is associated with the function T.

(b) Find the eigenvalues and eigenfunctions associated with T.

(c) The eigenfunction equation equation for R can be written

$$r^2 R'' + rR' - \lambda_n R = 0.$$

Solutions of this type of equation have the form $R(r) = r^\mu$ for certain values of μ. Plug this solution into the differential equation and solve for μ.

(d) We only want to consider bounded solutions of the PDE, so eliminate solutions from part (c) which are not bounded at $r = 0$.

(e) Combining the eigenfunctions from part (b) and part (d), write the solution of the PDE as an infinite series. Do not calculate the Fourier coefficients.

Helpful formulae:

$$\int \sin x \sin \beta x dx = \frac{\sin((1-\beta)x)}{2(1-\beta)} - \frac{\sin((1+\beta)x)}{2(1+\beta)}$$

$$\int \sin x \cos \beta x dx = -\frac{\cos((1-\beta)x)}{2(1-\beta)} - \frac{\cos((1+\beta)x)}{2(1+\beta)}$$

$$\int x \sin(\beta x) dx = \frac{\sin(\beta x) - \beta x \cos(\beta x)}{\beta^2}$$

$$\int x \cos(\beta x) dx = \frac{\cos(\beta x) + \beta x \sin(\beta x)}{\beta^2}$$

$$\Delta_2 u(r,\theta) = \frac{1}{r}\left(r u_r\right)_r + \frac{1}{r^2} u_{\theta\theta}$$

$$\Delta_3 u(\rho,\theta,\phi) = \frac{1}{\rho^2}\left(\rho^2 u_\rho\right)_\rho + \frac{1}{\rho^2}\left[u_{\phi\phi} + \frac{\cos\phi}{\sin\phi} u_\phi + \frac{1}{\sin^2\phi} u_{\theta\theta}\right]$$