Math 3213: Final Exam
April 15, 2011

Note: This exam allows the use of the textbook, but no other material.

(20%) 1. Answer the following questions using a short calculation or explanation.

(a) Let $S = \{(x, y)|y \geq 0\}$. Determine whether S is a subspace of $\mathbb{R}^2$.

(b) What is the length of the vector $\vec{w} = (-1 + 2i, -i, 4 + 3i) \in \mathbb{C}^3$?

(c) Let $R \in \mathcal{L}(\mathbb{C}^4)$ be defined as follows:

$$R(a, b, c, d) = (a + 2b + 3d, -b + c + 2d, 2a + b - c, 3a + 2c + d).$$

Is $\vec{v} = (1, i, -i, -1)$ an eigenvector of R? If so, what is the associated eigenvalue?

(d) Given an example of a function $f : \mathbb{C}^2 \to \mathbb{C}$ such that $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{C}^2$, but f is not linear.

(e) Let $T \in \mathcal{L}(\mathbb{R}^2)$ be given by a rotation $90°$ counter-clockwise, followed by a reflection in the x-axis. Find the matrix of T with respect to the standard basis.

(10%) 2. Let $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$. Prove

$$v_1 + \cdots + v_n \leq \sqrt{n}\left(v_1^2 + \cdots + v_n^2\right)^{\frac{1}{2}}.$$

(20%) 3. Consider the list $\mathcal{C} = ((1, 0, 1, 0), (0, 1, 0, 1))$ of vectors in $\mathbb{R}^4$.

(a) Show that $\mathcal{C}$ is linearly independent.

(b) Extend $\mathcal{C}$ to a basis $\mathcal{B} = ((1, 0, 1, 0), (0, 1, 0, 1), \vec{v}_3, \vec{v}_4)$ for $\mathbb{R}^4$.

(c) Use the Gram-Schmidt process on $\mathcal{B}$ to find an orthonormal basis for $\mathbb{R}^4$. (Note: You are NOT allowed to reorder the list.)

(10%) 4. Suppose V is a real finite dimensional vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T has even dimension.

(20%) 5. Suppose V is a finite dimensional real vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if there is an inner product on V that makes T into a self-adjoint operator.

(20%) 6. Let V be a non-zero finite dimensional real vector space and let $D : V \times V \to \mathbb{R}$ have the following properties for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $\alpha, \beta \in \mathbb{R}$.

1. $D(\alpha\vec{u} + \beta\vec{v}, \vec{w}) = \alpha D(\vec{u}, \vec{w}) + \beta D(\vec{v}, \vec{w})$
2. $D(\vec{u}, \vec{v}) = -D(\vec{v}, \vec{u})$

(a) Prove $D(\vec{u}, \vec{u}) = 0$.

(b) Prove $D(\vec{u} + \alpha\vec{v}, \vec{v}) = D(\vec{u}, \vec{v})$.

(c) Prove $D(\vec{w}, \alpha\vec{u} + \beta\vec{v}) = \alpha D(\vec{w}, \vec{u}) + \beta D(\vec{w}, \vec{v})$

(d) Let The dimenion of V be 3, and let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be an orthonormal basis for V. Define $D(\vec{e}_i, \vec{e}_j) = 1$ whenever $i < j$. Find a formula for $D(\vec{u}, \vec{v})$, for $\vec{u}, \vec{v} \in V$.