Math 3713 : Introduction to Analysis
Final Exam (December 19, 2006)

1. [**1 mark each**] Give an example of
   (a) a Cauchy sequence
   (b) a compact set
   (c) a conditionally convergent infinite series
   (d) a function $f : \mathbb{R} \to \mathbb{R}$ which is bounded, and has a limit at every point except, $x = 3$.

2. [**6 marks each**] Define
   (a) the power series generated by the sequence $\{a_n\}_{n=0}^{\infty}$.
   (b) Cauchy product
   (c) connect set
   (d) uniform continuity of a function f on domain D
   (e) a subsequence of a sequence, $\{a_n\}_{n=1}^{\infty}$.
   (f) equivalent sets, A and B

3. [**10 marks each**] State clearly and concisely each of the following:
   (a) the Modified Well-Ordering Principle
   (b) the Heine-Borel Theorem
   (c) the Alternating Series Test
   (d) a result relating a sequence and its subsequences with respect to convergence.
   (e) a result relating the limit of a function f at x_0 and sequences converging to x_0.

4. [**10 marks**] Use induction to prove $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

5. [**5 marks**] Find the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$.

6. [**10 marks**] Let $E_1, \cdots, E_n$ be compact sets. Prove $\displaystyle\bigcup_{i=1}^{n} E_i$ is compact. Give an example to show that the infinite union of compact sets need not be compact.

7. [**10 marks**] Suppose $\displaystyle\sum_{n=0}^{\infty} a_n$ and $\displaystyle\sum_{n=0}^{\infty} b_n$ converge. Prove that $\displaystyle\sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)$ converges to $\displaystyle\alpha\sum_{n=0}^{\infty} a_n + \beta\sum_{n=0}^{\infty} b_n$.

8. [**5 marks**] Evaluate $\displaystyle\lim_{x\to 9} \frac{x^2 - 81}{\sqrt{x} - 3}$. (Hint: let $u = \sqrt{x}$)

9. [**5 marks**] Define $f : (0,1) \to \mathbb{R}$ by $f(x) = \dfrac{1}{\sqrt{x}} - \sqrt{\dfrac{x+1}{x}}$. Can one define $f(0)$ to make f continuous at 0? Explain.

10. [**4 marks each**] Determine if each of the series is absolutely convergent, conditionally convergent, or divergent.
    (a) $\displaystyle\sum^{\infty} (-1)^n \ln n$