

Math 6132: Final Exam

April 9, 2011

- (10%) 1. Solve the following PDE: $x^{-1}u_x - y^2u_y = \sqrt{u}$ which passes through the curve $y = x^{-2}$, $u = 0$.

- (10%) 2. Find the regions in the xy -plane where the equation

$$(x+1)u_{xx} + 2yu_{xy} - (1-x)u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch the regions.

- (20%) 3. Use the method of factoring to solve

$$\left(\frac{\partial^3}{\partial x^3} - 3 \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial t} - 4 \frac{\partial}{\partial x} \frac{\partial^2}{\partial t^2} + 12 \frac{\partial^3}{\partial t^3} \right) u(x, t) = 0$$
$$u(x, 0) = 0, \quad u_t(x, 0) = x, \quad u_{tt}(x, 0) = x^2$$

(Hint: $\frac{\partial}{\partial x} - 3\frac{\partial}{\partial t}$ is a factor of the linear operator. Also, when solving for the arbitrary functions, integrate over the interval $[0, x]$.)

- (20%) 4. Solve:

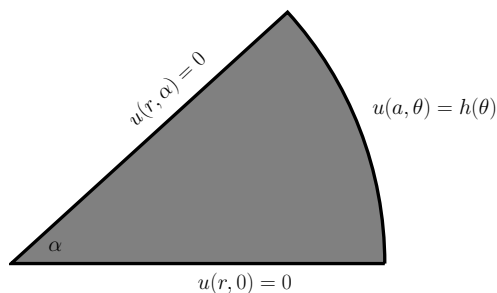
$$\begin{aligned} u_t &= 2u_{xx}, & 0 < x < \pi, \quad 0 < t < \infty \\ u_x(0, t) &= u_x(\pi, t) = 0 \\ u(x, 0) &= \sin x & 0 < x < \pi \end{aligned}$$

- (20%) 5. Find all eigenvalues and associated eigenfunctions of $-\frac{d^2}{dx^2}$ with the following boundary conditions: $X'(0) + X(0) = X'(1) + X(1) = 0$.

(20%) 6. Consider the following PDE

$$\begin{aligned}\Delta_2 u(r, \theta) &= 0 && \text{for } 0 < r < a, \quad 0 < \theta < \alpha \\ u(r, 0) &= u(r, \alpha) = 0 \\ u(a, \theta) &= h(\theta)\end{aligned}$$

as illustrated in the following diagram. Note that we are using polar coordinates.



- (a) Assume that $u(r, \theta) = R(r)T(\theta)$. Use separation of variables to generate ordinary differential equations for R and T . Make sure the negative sign is associated with the function T .
- (b) Find the eigenvalues and eigenfunctions associated with T .
- (c) The eigenfunction equation for R can be written

$$r^2 R'' + r R' - \lambda_n R = 0.$$

Solutions of this type of equation have the form $R(r) = r^\mu$ for certain values of μ . Plug this solution into the differential equation and solve for μ .

- (d) We only want to consider bounded solutions of the PDE, so eliminate solutions from part (c) which are not bounded at $r = 0$.
- (e) Combining the eigenfunctions from part (b) and part (d), write the solution of the PDE as an infinite series. Do not calculate the Fourier coefficients.

Helpful formulae:

$$\int \sin x \sin \beta x dx = \frac{\sin((1-\beta)x)}{2(1-\beta)} - \frac{\sin((1+\beta)x)}{2(1+\beta)}$$

$$\int \sin x \cos \beta x dx = -\frac{\cos((1-\beta)x)}{2(1-\beta)} - \frac{\cos((1+\beta)x)}{2(1+\beta)}$$

$$\int x \sin(\beta x) dx = \frac{\sin(\beta x) - \beta x \cos(\beta x)}{\beta^2}$$

$$\int x \cos(\beta x) dx = \frac{\cos(\beta x) + \beta x \sin(\beta x)}{\beta^2}$$

$$\Delta_2 u(r, \theta) = \frac{1}{r} (ru_r)_r + \frac{1}{r^2} u_{\theta\theta}$$

$$\Delta_3 u(\rho, \theta, \phi) = \frac{1}{\rho^2} (\rho^2 u_\rho)_\rho + \frac{1}{\rho^2} \left[u_{\phi\phi} + \frac{\cos \phi}{\sin \phi} u_\phi + \frac{1}{\sin^2 \phi} u_{\theta\theta} \right]$$