Math 2013: Lab 01

Project: The Geometry of Euclidean Four-Space [1]

Congratulations! You have just obtained a contract as architect of a four-dimensional office. In order to figure out how to arrange the furniture, where to put the closets, etc. you naturally will have to explore the geometry of four-dimensional space. We treat this in a fashion analagous to what we have in two and three dimensions and in order to answer the the questions you should use your knowledge of $\mathbb{R}^2$ and $\mathbb{R}^3$. The main difference is that there are three-dimensional "flat" objects which are unbounded, and which do not fill the whole space. These are called hyperplanes.

1. (a) Find the general equation of a hyperplane in both vector form and in Cartisean (or general) form. (Hint: look at the corresponding equations for planes in three dimensions.)

   (b) Give an example of a pair of parallel hyperplanes, one of which goes through the origin.

2. (a) Intuitively, if two hyperplanes intersect, they should intersect in a plane. Use this intuition, and the analogy of two planes intersecting in a line, to find various forms of the equation of a plane in 4-space.

   (b) Give examples of: a pair of planes which intersect in a line, a pair of planes which do not intersect and are parallel, a pair of planes which intersect at only one point, and a pair of **skew planes** which do not intersect and are not parallel.

   (c) Just as we can have coplanar lines in 3-space, we can have planes which lie in the same hyperplane in 4-space. Give an example, and devise a test which allows you to tell whether a pair of planes is **co-hyperplanar**.

3. (a) Under what circumstances do four points determine a hyperplane? (Hint: they cannot be colinear, for one thing.)

   (b) A hypersphere is the set of all points in four-space equidistant from a given point. Write the equation of the hypersphere centered at $(0, 0, 0, 0)$ with radius 3, assuming the usual distance formula.

4. (a) Let R be the region in 4-space which meets the following conditions: for each point in R, the coordinates are alway non-negative, and the coordinates sum to 1. Find the hypervolume of the region R. (Hint: the analogous problem in 2-space is to find the area of the region with $x \geq 0$, $y \geq 0$, and $x + y = 1$.)

   (b) Show that the hypervolume of a hypersphere of radius r is $\dfrac{1}{2}\pi^2 r^4$.

Bonus:

1. The office itself will be a **right hyper-parallelopiped**, that is, a hyperbox. This is a bounded chunk of 4-space with "walls" consisting of hyperplanes meeting at right angles. Give an expression for the hypervolume of a general hyperbox. How many hyperplane "walls" are there? How many ordinary chalk boards do you need to put one on each plane along where the hyperwalls intersect?

2. At the last minute, your shipment of hyperlights is held up by an environmental impact assessment, and you are stuck with only ordinary three-dimensional lights (which you can think of as small spheres emitting light is a 3-dimensional sphere around each of them). How many will you need to light the office, assuming an open plan (that is, no walls)? How would you write the equation of an ordinary sphere centered at $(0, 0, 0, 0)$ with radius 3?

[1] With the exception of question 4 and the question about planes intersecting at a single point, this project is taken from the course Mathematics 2023 given at Acadia University in 1992.

Math 2013: Lab 02

Project: Kepler's Laws [2]

Johannes Kepler stated the following three laws of planetary motion on the basis of masses of data on the positions of planets at various times.

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Kepler formulated these laws because they fitted the astronomical data. He was not able to see why they were true or how they related to each other. But Sir Isaac Newton, in his *Principia Mathematica* of 1687, showed how to deduce Kepler's three laws from two of Newton's own laws, the Second Law of Motion ($\vec{F} = m\vec{a}$) and the Law of Universal Gravitation $\left(\vec{F} = -\dfrac{GMm}{r^3}\vec{r}\right)$. In Section 13.4 we proved Kepler's First Law using the calculus of vector functions. In this project we guide you through the proofs of Kepler's Second and Third Laws and explore some of their consequences.

1. Use the following steps to prove Kepler's Second Law. The notation is the same as in the proof of the First Law in Section 13.4. In particular, use polar coordinates so that $\vec{r} = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j}$, and $\vec{h} = \vec{r} \times \vec{v}$, with $h = |\vec{h}|$.

   (a) Show that $\vec{h} = r^2 \dfrac{d\theta}{dt}\vec{k}$.

   (b) Deduce that $r^2 \dfrac{d\theta}{dt} = h$.

   (c) If $A = A(t)$ is the area swept out by the radius vector $\vec{r} = \vec{r}(t)$ in the time interval $[t_0, t]$, show that
   $$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

   (d) Deduce that
   $$\frac{dA}{dt} = \frac{1}{2} h = \text{constant}.$$

   This says that the rate at which A is swept out is constant and proves Kepler's Second Law.

2. Let T be the period of a planet about the sun; that is, T is the time required for it to travel once around its elliptical orbit. Suppose that the lengths of the major and minor axes of the ellipse are $2a$ and $2b$ respectively.

   (a) Use part (1d) to show that $T = 2\pi ab/h$.

   (b) Show that $\dfrac{h^2}{GM} = ed = \dfrac{b^2}{a}$.

   (c) Use parts (2a) and (2b) to show that $T^2 = \frac{4\pi^2}{GM} a^3$.

   This proves Kepler's Third Law. (Notice that the proportionality constant $4\pi^2/(GM)$ is independent of the planet.)

3. The period of the earth's orbit is approximately 365.25 days. Use this fact and Kepler's Third Law to find the length of the major axis of the earth's orbit. You will need the mass of the sun, $M = 1.99 \times 10^{30}$ kg, and the gravitational constant, $G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2$.

4. It is possible to place a satellite into orbit about the earth so that it remains fixed above a given location on the equator. Compute the altitude that is needed for such a satellite. The earth's mass is 5.98×10^{24} kg; its radius is 6.37×10^6 m. (This orbit is called the Clarke Geosynchronous Orbit after Arthur C. Clarke, who first proposed the idea in 1945. The first such satellite, Syncom 2, was launched in July 1963.)

[2] This project is taken from Stewart, pp. 848-9.

Math 2013: Lab 03

Project: Vector Calculus in Curvilinear Coordinates

As we have seen previously, there are times when a change of coordinates simplifies the problem nicely. We have used polar coordinates to integrate over planar regions and cylindrical and spherical coordinates for integration over volumes.

This term we have studied the fundamental theorem for line integrals, which involves integration of the gradient of a function, and we will be considering integration of the divergence of vector fields. In this project we will investigate the gradient and divergence operators in cylindrical and in spherical coordinates, and then calculate the Laplacian in these coordinate systems.

1. Recall, in Cartesian coordinates, we have

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}. \tag{1}$$

   To compute the gradient in the various coordinate systems, we will first re-write the partial derivatives in terms of the new variables.

   (a) In cylindrical coordinates, r, θ, and z, use the chain rule to re-write $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

   (b) Substitute these new expressions into (1) and group the partial derivatives $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial \theta}$, and $\frac{\partial f}{\partial z}$.

   (c) If we define three new unit vectors,

$$\begin{aligned}\vec{e}_r &= \cos\theta\vec{i} + \sin\theta\vec{j}\\ \vec{e}_\theta &= -\sin\theta\vec{i} + \cos\theta\vec{j}\\ \vec{e}_z &= \vec{k}\end{aligned}$$

   we can conclude that

$$\nabla f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{\partial f}{\partial z}\vec{e}_z \tag{2}$$

   in cylindrical coordinates.

2. Use the above method to show that the gradient in spherical coordinates is

$$\nabla f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r}\frac{\partial f}{\partial \phi}\vec{e}_\phi \tag{3}$$

   with appropriate choices for $\vec{e}_r, \vec{e}_\theta$, and $\vec{e}_\phi$.

3. Let $\vec{F} = P\vec{e}_r + Q\vec{e}_\theta + R\vec{e}_z$ be a vector field in cylindrical coordinates. We are told that the divergence of $\vec{F}$ is

$$\nabla \cdot \vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rP) + \frac{1}{r}\frac{\partial Q}{\partial \theta} + \frac{\partial R}{\partial z}. \tag{4}$$

   Use (2) and (4) to show

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

4. Let $\vec{F} = P\vec{e}_r + Q\vec{e}_\theta + R\vec{e}_\phi$ be a vector field in spherical coordinates. We are told that the divergence of $\vec{F}$ is

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P\right) + \frac{1}{r\sin\phi}\frac{\partial Q}{\partial \theta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi). \tag{5}$$

   Use (3) and (5) to show

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{1}{r^2\tan\phi}\frac{\partial f}{\partial \phi}.$$

Math 2013: Lab 04

Project: Exterior Calculus in $\mathbb{R}^3$

For this project we are going to investigate differential forms. Differential forms, according to H. Flanders "are the things which occur under integral signs." [3] For example, a line integral, $\int_C Pdx + Qdy + Rdz$ gives the 1-form $\mu = Pdx + Qdy + Rdz$, and a surface integral, $\iint_S Adxdy + Bdydz + Cdzdx$ gives the 2-form $\nu = Adxdy + Bdydz + Cdzdx$, while a volume integral, $\iiint_E Fdxdydz$ give the 3-form $\omega = Fdxdydz$. We define 0-forms to be functions.

There are three main algebraic operations on the space of differential forms: scalar multiplication and addition which are performed in the intuitive manner (ie. component-wise), and the exterior (or wedge) product, which we will define later. There is also an exterior derivative operator, denoted d, and another operator, called the Hodge star, denoted $*$. With these we will describe an exterior calculus.

1. Let $\alpha = xdx + yzdz$ and $\beta = y^2dx + yzdy - xdz$. Compute $x\alpha$ and $2\alpha + z\beta$.

2. The exterior product of dx^i and dx^j is $dx^i \wedge dx^j$ and it obeys the rule $dx^i \wedge dx^j + dx^j \wedge dx^i = 0$. (ie. it is **anti-commutative**) Also, we note that $(adx^i) \wedge (bdx^j) = (ab)dx^i \wedge dx^j$. The wedge product is **distributive**: $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$, and it is **associative**: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.
   (a) For α and β as in question 1, compute $\alpha \wedge \beta$, $\alpha \wedge \alpha$, and $\beta \wedge (\alpha \wedge \beta)$.
   (b) Let $\omega = Pdx + Qdy + Rdz$ and $\nu = Adx + Bdy + Cdz$. Compute $\omega \wedge \nu$ and collect the terms in the following order: $dy \wedge dz, dz \wedge dx$, and $dx \wedge dy$. Compare the result to the cross-product: $\langle P, Q, R\rangle \times \langle A, B, C\rangle$.

3. In $\mathbb{R}^3$, the space of 0-forms and the space of 3-forms can be thought of as having dimension 1, while the space of 1-forms and the space of 2-forms can be said to have dimension 3. Thus there is a linear mapping from 0-forms to 3-forms, and from 1-forms to 2-forms. This mapping is called the Hodge star isomorphism, and is defined as follows:

$$\begin{aligned} *1 &= dx \wedge dy \wedge dz \\ *dx &= dy \wedge dz \\ *dy &= dz \wedge dx \\ *dz &= dx \wedge dy \end{aligned}$$

   Note, that for $\mathbb{R}^3$, $**\omega = \omega$, so this defines the isomorphism for 2- and 3-forms.
   (a) For α and β as is question 1, compute $*\alpha$, $\alpha \wedge *\alpha$, and $*(\beta \wedge *\beta)$.
   (b) Let ω and ν be as is question 2 (b). Compute $*(\omega \wedge \nu)$ and $*(\omega \wedge *\nu)$, and identify these with the cross product and the dot product of vectors.

4. The exterior derivative, d, takes p-forms to $(p+1)$ forms, and is defined for $\mathbb{R}^3$ as follows:

$$df = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x^i}\right) dx^i$$

$$d\omega = \sum_{i=1}^{3} \left(\frac{\partial \omega}{\partial x^i}\right) \wedge dx^i$$

$$\text{CORRECTION}: \ d\omega = \sum_{i=1}^{3} dx^i \wedge \left(\frac{\partial \omega}{\partial x^i}\right)$$

   where $\partial\omega/\partial x^i$ acts on the coefficients of the terms in ω. **All solutions are for the uncorrected definition.**
   (a) For α and β as is question 1, compute $d\alpha$, $d\beta$, and $d(\alpha \wedge \beta)$.
   (b) If a 0-form f has continuous second order partial derivatives, prove: $d^2 f = d(df) = 0$.
   (c) For a 0-form f, a 1-form ω, and a 2-form ν, compute df, $*(d\omega)$, and $*(d\nu)$ and identify these with the operators from vector calculus: gradient, divergence, and curl.

[3] Harley Flanders, *Differential Forms with Applications to the Physical Sciences*, p. 1

Math 2013: Lab 05

Project: Integral Theorems

Note: for this project, you may assume that all the appropriate conditions for the relevant theorems apply.

1. Use the vector form of Green's Theorem, $\oint_C \vec{F} \cdot \vec{n} ds = \iint_D \nabla \cdot \vec{F} dA$, to prove:

$$\iint_D f \nabla^2 g dA = \oint_C f \nabla g \cdot \vec{n} ds - \iint_D \nabla f \cdot \nabla g dA, \tag{6}$$

   where $D \subset \mathbb{R}^2$ and $C = \partial D$, with a positive orientation.

2. Use (6) to prove

$$\iint_D \left(f \nabla^2 g - g \nabla^2 f \right) dA = \oint_C \left(f \nabla g - g \nabla f \right) \cdot \vec{n} ds. \tag{7}$$

3. A function g is call **harmonic** on D if $\nabla^2 g = 0$ on D. If g is harmonic on D, prove $\oint_{\partial D} D_{\vec{n}} g ds = 0$, recalling that $D_{\vec{u}}$ indicates the directional derivative.

4. If f is harmonic on D a simple region in $\mathbb{R}^2$, show that the line integral $\int_C f_y dx - f_x dy$ is independent of path in D.

5. Prove that

$$\iint_S f \vec{n} dS = \iiint_E \nabla f dV$$

   where the integration is done component-wise.

6. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity. The total buoyant force on the solid due to the pressure distribution is given by

$$\vec{F} = -\iint_S p \vec{n} dS$$

   where $\vec{n}$ is the outward normal. Show that $\vec{F} = -W\vec{k}$, where W is the weight of the liquid displaced by the solid. The result is **Archimedes' principle**: The buoyant force on an object equals the weight of the displaced liquid.

Math 2013: Lab 06

Project: Series Solutions of Ordinary Differential Equations

Sometimes solutions to differential equations cannot be expressed in terms of "elementary functions" (that is, in terms of polynomials, exponential, trigonometric, logarithmic, etc. functions). But these differential equations can still define functions. One method of solving these equations is to assume the solution have the form:

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots, \tag{8}$$

where the coefficients are determined by the differential equation.

1. Consider the equation, $y' - y = 0$. We know the solution is $y = Ae^x$. Use (8) to find the series expansion of e^x.
   (a) For (8), find y' by differentiating term by term.
   (b) Write $y' - y$ and collect like powers of x and find a formula for the coefficient of x^n in this expression.
   (c) Since $y' - y = 0$, all the coefficients must zero. Find a recursion relation for the n^{th} coefficient. (See for example equation 6, p. 1134).
   (d) Find a formula for c_n in terms of c_0, and substitute this back into (8).
2. The equation $x^2y'' + xy' + x^2y = 0$ is known as Bessel's equation of order 0. Use the following steps to find a series solution of this equation, with coefficients in terms of c_0 and c_1.
   (a) From (8) find the series for y' and y''.
   (b) Find x^2y'', xy', and x^2y by multiplying each term in the respective series by either x^2 or x, depending on the situation.
   (c) Write out explicitly the first five non-zero terms of each series: x^2y'', xy', and x^2y.
   (d) Find a recursive relation for the coefficients of x^n.
   (e) Consider even and odd coefficients separately, and find a formula for c_{2n} in terms of c_0 and for c_{2n+1} in terms of c_1.
   (f) Write $y = c_0 J_0(x) + c_1 Y_0(x)$. The functions J_0 and Y_0 are called Bessel function of the first and second kind, of order 0. Give the series expression for $J_0(x)$ and $Y_0(x)$.

   You may find useful the following formula:

$$1 \cdot 2 \cdot 3 \cdots n = n!$$
$$2 \cdot 4 \cdot 6 \cdots 2n = 2^n n!$$
$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{(2n)!}{2^n n!}$$

Math 2013: Lab 07

Project: Introduction to Analysis

Material from Appendix A of the textbook may be of use for this project.

1. Prove: If $\{a_n\}_{n=1}^{\infty}$ converges to A and to B, then $A = B$. (Hint: Try proof by contradiction. Assume $A \neq B$ and show there is an inconsistent result.)

2. Prove: If $\{a_n\}_{n=1}^{\infty}$ converges to A, then $\{a_n\}_{n=1}^{\infty}$ is bounded. (Hint: Let ϵ be a fixed number, like $\epsilon = 1$.)

3. Prove: If $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}_{n=1}^{\infty}$ converges to B, then $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $A + B$.

4. A sequence $\{a_n\}_{n=1}^{\infty}$ is **Cauchy** if and only if for each $\epsilon > 0$ there is a positive integer N such that if $n, m \geq N$ then $|a_n - a_m| < \epsilon$. Prove: Every convergent sequence is Cauchy. (Hint: let $\epsilon > 0$ and consider $\epsilon' = \epsilon/2 > 0$ for the usual definition of convergence.)

5. Prove: If $c > 1$, then $\{\sqrt[n]{c}\}_{n=1}^{\infty}$ converges to 1. (Hint: Show the sequence is decreasing and that it is bounded below by 0.) For additional work, prove: if $0 < c < 1$ then $\{\sqrt[n]{c}\}_{n=1}^{\infty}$ converges to 1.

Math 2013: Lab 08

Project: Introduction to Fourier Series

1. Consider the following differential equation: $-X''(x) = \lambda X(x)$, where $\lambda \in \mathbb{R}$.
   (a) Let $\lambda = \beta^2$, where $\beta > 0$. Write the general solution of the differential equation. Use A and B for the arbitrary coefficients.
   (b) In the above solution, there are three unknowns: A, B, and β. Use the boundary conditions $X(0) = X(\pi) = 0$ to solve for two of the unknowns. One of the coefficients, A or B, will be left unknown.
   (c) From the above, we have limited the values for λ. These will form a sequence. Write a formula for the n^{th} term, denoted λ_n.
   (d) Also from above, for each λ_n, there is a function $X_n(x)$. In this case the arbitrary coefficient may be assumed to be 1. Find a formula for $X_n(x)$. The numbers λ_n are called eigenvalues, and the functions $X_n(x)$ are called eigenfunctions.

2. The context for the above question comes from partial differential equations (PDEs). From work done in PDEs, we know that for these problems, solutions may be summed. So we will now consider functions $f(x) = \sum_{n=1}^{\infty} A_n X_n(x)$.
   (a) Let $n, m \in \mathbb{N}$. Prove: $\int_0^\pi \sin nx \sin mx dx = 0$ if $m \neq n$, and $\int_0^\pi \sin^2 nx dx = \frac{\pi}{2}$.
   (b) If $f(x) = \sum_{n=1}^{\infty} A_n \sin nx$, prove that $A_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$. These coefficients are called Fourier coefficients, and the series is called the Fourier (sine) series for $f(x)$ on $(0, \pi)$.
   (c) Find the Fourier coefficients for $f(x) = 1$, and write the Fourier series for $f(x)$. Note: If $x = \frac{\pi}{2}$, we get a "nice" series whose sum is π.

3. Consider the following differential equation: $-X''(x) = \lambda X(x)$, where $\lambda \in \mathbb{R}$, with the boundary conditions $X(0) = X(\pi) = 0$. Assume that $\lambda = -\beta^2$ where $\beta > 0$, and show that there cannot be any eigenfunctions in this case. (Note: $X(x) = 0$ is not allowed to be an eigenfunction.)

Math 2013: Lab 09

Project: Problems Plus – Sequences and Series

1. p. 761, # 3
2. p. 761, # 5
3. p. 762, # 13
4. p. 762, # 15
5. p. 763, # 16

Math 2013: Lab 10

Project: Fixed Points in Population Growth

Population models sometimes use the method of difference equations to predict future population growth. This method uses discrete time steps, and at each interval, the population depends on the previous population size, that is, $P_{n+1} = f(P_n)$ for some function f. In this lab, we will explore fixed points in population models. These are population sizes that stay unchanged as time progresses. Mathematically, we are looking for population sizes, P, such that $P = f(P)$.

1. For this lab, we will consider the model $f(x) = rx(1-x)$, where r is a positive real number. We will later show that $0 \le r \le 4$.
   (a) Show that this model has two fixed point ($x = f(x)$) when $r > 1$. One fixed point is $x = 0$, the other is a function of r. We will denote this fixed point by $x^*(r)$, or just x^*.
   (b) Graph on the same graph, $y = x$ and $y = f(x)$. Label the fixed points.
   (c) Linearize (find the equation of the tangent line) the function f at the fixed point x^*. Use the point-slope form. Use this equation to approximate $f(P_n)$. Recall that $f(P_n) = P_{n+1}$ and $f(x^*) = x^*$. Define $u_n = P_n - x^*$. Show that the linearization can be written as $u_{n+1} = f'(x^*)u_n$. What happens if $|f'(x^*)| > 1$? $|f'(x^*)| < 1$? Find the value(s) r such that $f'(x^*(r)) = -1$.
   (d) Compute (using OpenOffice.org Calc, or some other spreadsheet program, like Gnumeric) the first 100 iterates $(P_1, P_2, \ldots, P_{100})$ with
      i. $r = 1.2$ and $P_0 = 0.33$
      ii. $r = 2.5$ and $P_0 = 0.33$
      iii. $r = 3.2$ and $P_0 = 0.67$
      iv. $r = 3.5$ and $P_0 = 0.67$

      Plot these four sequences (as an X-Y scatter plot), letting n be the horizontal axis.
2. A period two fixed point is a solution of the equation $f(f(x)) = x$.
   (a) Explain why the equation $f(x) = f^{-1}(x)$ could be used to find period two fixed points. Using the same f as is the first question, for r value of $1, 2$, and 3.5, plot $f(x)$ and $f^{-1}(x)$ on the same axis using the reflection method for the inverse function. Note the intersection of these two graphs.
   (b) Solve analytically for the period two fixed points. This requires you to factor a fourth degree polynomial. To aid in this, note that a fixed point is also a period two fixed point. For what values of r are there more than two period two fixed points?
   (c) What would be an interpretation of a period two fixed point?
3. It was stated at the beginning of the lab that $0 \le r \le 4$. Show that if $r > 4$, then the maximum of $f(x)$ is greater than 1. This would mean that the iterations could take us outside the "domain" of the model.