Math 431: Metric Spaces and Topology Assignments

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Assignment 1: Due January 27, 2017

2.3: Suppose that V, X, Y are sets with $V \subseteq X \subseteq Y$ and suppose that U is a subset of Y such that $X \setminus V = X \cap U$. Prove that

$$V = X \cap (Y \setminus U).$$

2.6: Suppose that for some set X and some indexing sets I, J we have $U = \bigcup_{i \in I} B_{i1}$ and $V = \bigcup_{j \in J} B_{j2}$ where each B_{i1}, B_{j2} is a subset of X. Prove that

$$U \cap V = \bigcup_{(i,j) \in I \times J} B_{i1} \cap B_{j2}.$$

3.4: Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x, 2x)$. Describe the following sets:

- $f([0, 1])$
- $f^{-1}([0, 1] \times [0, 1])$
- $f^{-1}(D)$ where $D = \{(x, y) | x^2 + y^2 \leq 1\}$.

3.9: Let $f : X \to Y$ be a map and $A \subseteq X, C \subseteq Y$. Prove that

(a) $f(A) \cap C = f(A \cap f^{-1}(C))$

(b) If also $B \subseteq X$ and $f^{-1}(f(B)) = B$ then $f(A) \cap f(B) = f(A \cap B)$.

4.3: Formulate and prove analogues of Exercises 4.1 and 4.2 for inf.

4.5: Show that there is no rational number q such that $q^2 = 2$.

4.8: Prove that between any two distinct real numbers there is an irrational number.

4.11: Given a set of r non-negative real numbers $\{a_1, \ldots, a_r\}$, let $a = \max a_i$. Prove that for any positive integer n,

$$a^n \leq a_1^n + \cdots + a_r^n \leq r a^n.$$

By taking n^{th} roots throughout, deduce that

$$a \leq (a_1^n + \cdots + a_r^n)^{1/n} \leq r^{1/n} a,$$

and hence that

$$\lim_{n \to \infty} (a_1^n + \cdots + a_r^n)^{1/n} = a.$$

Assignment 2: Due February 10, 2017

1. For $x = (-2, 1)$ and $y = (3, 4)$, compute the distance from x to y in each of the following metrics:

   (a) The discrete metric.

   (b) d_1, d_2, and d_∞ as described in Example 5.7

2. Describe pictorially (on a graph) the set of points $x \in \mathbb{R}^2$ whose distance from $(4, 2)$ is less than or equal to 1 with respect to the following metrics:

   (a) The discrete metric.

   (b) d_1, d_2, and d_∞ as described in Example 5.7

3. From geometry, a parabola is the collection of points which are equal distance from a fixed point and a given line. Sketch, with justification, the parabola defined by the point $(0, 1)$ and the line $y = -1$ using the metric d_1 defined in Example 5.7.

4. Prove that $d(x, y) = |e^x - e^y|$ defines a metric on $\mathbb{R}$. Describe the set of points in $\mathbb{R}$ whose distance from 1 is at most 5 under this metric.

5. Let X be a metric space with metric d. Show that $D : X \times X \to \mathbb{R}$ defined by

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

   is also a metric on X. Also show that X is a bounded set in the metric space (X, D).

5.2: Given points x, y, z, t in a metric space (X, d), prove that

$$|d(x, y) - d(z, t)| \leq d(x, z) + d(y, t)$$

5.5: Suppose that x, y are distinct points in a metric space (X, d) and let $\epsilon = d(x, y)/2$. Prove that $B_\epsilon(x)$ and $B_\epsilon(y)$ are disjoint.

5.18: Suppose that in a metric space X we have $B_s(x) = B_r(y)$ for some $x, y \in X$ and some positive real numbers r, s. Is $x = y$? Is $r = s$?

6. Let (X, d) be a metric space, and let P be the set of all non-empty subsets of X, that is $P = \{A | A \subseteq X, A \neq \emptyset\}$. We can define the distance between two elements of P by $D(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$. Then (P, D) is not a metric space. In what ways does it fail? Give an example of each type of failure.

7. Let $A = \{(x, y) \in \mathbb{R}^2 | 0 \leq x, y\}$. Determine whether A is open, closed, or neither with respect to: the discrete metric, d_1, d_2 and d_∞.

5.7: Show that if S is a bounded set in $(\mathbb{R}^n, d_2)$, then S is contained in

$$[a, b] \times [a, b] \times \cdots \times [a, b]$$

for some $a, b \in \mathbb{R}$.

5.14: Show that for any $x, y \in \mathbb{R}^n$,

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y) \le n d_\infty(x,y).$$

6.3: Prove that any finite subset of a metric space X is closed in X.

6.16: For a point x and a non-empty subset A of a metric space (X,d), define $d(x,A) = \inf\{d(x,a) | a \in A\}$.

(a) Prove that $d(x,A) = 0$ iff $x \in \overline{A}$.

(b) Show that if y is another point in X, then $d(x,A) \le d(x,y) + d(y,A)$.

(c) Prove that $x \mapsto d(x,A)$ gives a continuous map from X to $\mathbb{R}$.

Assignment 3: Due February 24, 2017

6.18: Prove that a finite subset of a metric space has no limit points.

6.23: For a subset A of a metric space X, prove:

(a) $A^\circ = A \setminus \partial A = \overline{A} \setminus \partial A$,

(b) $\overline{X \setminus A} = X \setminus A^\circ$,

(c) $\partial A = \overline{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$,

(d) ∂A is closed in X.

6.27: Prove that the metrics $d^{(2)}$, $d^{(3)}$ in Exercise 5.12 are topologically equivalent to d.

7.2: Give an example of two topologies $\mathcal{T}_1, \mathcal{T}_2$ on the same set such that neither contains the other.

7.3: Show that the intersection of two topologies on the same set X is also a topology on X, but that their union may or may not be a topology. Does this first result extend to the intersection of an arbitrary family of topologies on X?

7.4: Prove that we get a topology for $\mathbb{N} = \{1, 2, 3, \ldots\}$ by taking the open sets to be $\emptyset, \mathbb{N}$ and $\{1, 2, 3, \ldots, n\}$ for each $n \in \mathbb{N}$.

7.6: Let $\mathcal{T}$ be the collection of all subsets of $\mathbb{R}$ consisting of $\emptyset, \mathbb{R}$ together with all intervals of the form $(-\infty, b)$. Show that $\mathcal{T}$ is a topology for $\mathbb{R}$.

8.1: Let $f : X \to Y$ be a map of topological spaces. Prove that f is continuous in the following cases:

(a) $X = Y$ and f is the identity map.

(b) f is a constant map.

(c) $\mathcal{T}_X$ is discrete.

(d) $\mathcal{T}_Y$ is indiscrete.

8.5: Prove that the set of all open intervals $\{(a,b) : a,b \in \mathbb{R}, a < b\}$ is a basis for the usual topology on $\mathbb{R}$.

9.5: Give either a proof of, or a counterexample to, each of the following:

(a) If $f : X \to Y$ is a continuous map of topological spaces and $A \subseteq X$ is closed, then $f(A) \subseteq Y$ is closed.

(b) If A is open in a topological space X and $B \subseteq X$ then $A \cap \overline{B} = \overline{A \cap B}$.

(c) If $f : X \to Y$ is a continuous map of topological spaces and $B \subseteq Y$ then $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

9.7: Prove that a map $f : X \to Y$ of topological spaces is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$.

9.11: Let $A_1, \ldots, A_m$ be subsets of a topological space X. Prove that the interior of $\bigcap_{i=1}^m A_i$ equals $\bigcap_{i=1}^m \mathring{A}_i$.

9.15: Let A be a subspace of a topological space X. Show that $\overline{A}$ is the disjoint union of $\mathring{A}$ and ∂A. Deduce that if B is another subspace of X such that $B \cap A \neq \emptyset$ then either $B \cap \partial A \neq \emptyset$ or $B \cap \mathring{A} \neq \emptyset$.

10.5: Suppose that $(A, \mathcal{T}_A)$ is a subspace of a space $(X, \mathcal{T})$ and that $V \subseteq X$ is closed in X. Prove that $V \cap A$ is closed in $(A, \mathcal{T}_A)$.

10.12: Suppose that $\mathcal{S}$ is the Seirpinski space of Example 7.7. Find the product topology $\mathcal{S} \times \mathcal{S}$.

Assignment 4: Due March 17, 2017

10.15: (a) Prove that if W is open in a topological product $X \times Y$ then $p_X(W)$ is open in X and $p_Y(W)$ is open in Y.

(b) Give an example of a closed set $W \subset \mathbb{R} \times \mathbb{R}$ whose projection $p_1(W)$ on the x-axis is not closed in $\mathbb{R}$.

10.16: Suppose that X, Y are spaces and that $A \subseteq X, B \subseteq Y$. Prove that

(a) the interior of $A \times B$ is $\mathring{A} \times \mathring{B}$.

(b) $\overline{A \times B} = \overline{A} \times \overline{B}$.

(c) $\partial(A \times B) = ((\partial A) \times \overline{B}) \cup (\overline{A} \times (\partial B))$.

11.4(c,d): Prove Proposition 11.7 (c,d)

(c): If $f : X \to Y$ is an injective continuous map of topological spaces and Y is Hausdorff, then so is X.

(d): If X and Y are homeomorphic then X is Hausdorff iff Y is Hausdorrf. In other words, Hausdorffness is a topological property.

11.5: Suppose that $f : X \to Y$ is a continuous map of a topological space X to a Hausdorff space Y. Prove that the graph G_f of f is a closed subset of the topological product $X \times Y$.

11.6: (a) Prove that if x is any point in a Hausdorff space X, then the intersection of all open subsets of X containing x is $\{x\}$.

(b) Give an example to show that the conclusion of (a) does not imply that X is Hausdorff. [**Hint: Think about the co-finite topology on an infinite set.**]

11.8: Suppose that X, Y are spaces, with Y Hausdorff, and that A is a subspace of X. Prove that if $f, g : \overline{A} \to Y$ are continuous and $f(a) = g(a)$ for all $a \in A$ then $f = g$.

Assignment 5: Due April 7, 2017

12.5: Suppose that for each $i \in \{1, 2, \ldots, n\}$ that A_i is a connected subset of a space X, such that $A_i \cap A_{i+1} \neq \emptyset$ for each $i \in \{1, 2, \ldots, n-1\}$. Prove that $\bigcup_{i=1}^n A_i$ is connected. Does this result extend to an infinite sequence (A_i) of connected subsets?

12.11: Give either a proof of, or a counterexample to, each of the following.

(a) Suppose that X, Y are spaces with subsets A, B. Suppose that neither $X \setminus A$ nor $Y \setminus B$ is connected. Then $(X \times Y) \setminus (A \times B)$ is not connected.

(b) Suppose that A, B are subsets of a space X and that both $A \cap B$ and $A \cup B$ are connected. Then A and B are connected.

(c) Suppose that A, B are closed subsets of a space X and that both $A \cap B$ and $A \cup B$ are connected. Then A and B are connected.

12.14: Prove Example 12.22(b), that an annulus in $\mathbb{R}^2$, (that is, a set of the form $\{(x, y) \in \mathbb{R}^2 | a \leq (x-c)^2 + (y-d)^2 \leq b\}$ for some real numbers a, b, c, d with $0 < a < b$) is path connected.

12.19: Give an example of a sequence of closed connected subsets V_n of the Euclidean plane such that $V_n \supseteq V_{n+1}$ for each $n \in \mathbb{N}$ but $\bigcap_{n=1}^\infty V_n$ is not connected.

13.4: Which of the following subsets of $\mathbb{R}$, $\mathbb{R}^2$ are compact?

(a) $[0, 1)$

(b) $[0, \infty)$

(c) $\mathbb{Q} \cap [0, 1]$

(d) $\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$

(e) $\{(x,y) \in \mathbb{R}^2 | \, |x| + |y| \leq 1\}$

(f) $\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$

(g) $\left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x \geq 1, 0 \leq y \leq \frac{1}{x}\right\}$

13.5: Given topologies $\mathcal{T}, \mathcal{T}'$ on a set X with $\mathcal{T} \subseteq \mathcal{T}'$, prove that if $(X, \mathcal{T}')$ is compact, then so is $(X\mathcal{T})$.

13.13: Suppose that X is a compact Hausdorff space and suppose that $f : X \to X$ is a continuous map. Let $X_0 = X, X_1 = f(X_0)$ and inductively define $X_{n+1} = f(X_n)$ for all $n \geq 1$.

(a) Show that $A = \bigcap_{n=0}^{\infty} X_n$ is non-empty. [Hint: Remember Exercise 13.11.]

(b) Show further that $F(A) = A$. [Hint: To show that $a \in A$ is in $f(A)$ apply Exercise 13.11 to the sets $V_n = f^{-1}(a) \cap X_n$.]

13.14: Suppose that X is a compact metric space with metric d and that $f : X \to X$ is a continuous map such that for every $x \in X$, $f(x) \neq x$. Prove that there exists $\epsilon > 0$ such that $d(f(x), x) \geq \epsilon$ for all $x \in X$. [Hint: Show that the map $g : X \to \mathbb{R}$ defined by $g(x) = d(f(x), x)$ is continuous so it attains its bounds.]

13.22: For any topological space $(X, \mathcal{T})$, let $X' = X \cup \{\infty\}$ where ∞ is any object not in X. Let $\mathcal{T}'$ be the union of $\mathcal{T}$ with all sets of the form $V \cup \{\infty\}$ where $V \subseteq X$ and $X \setminus V$ is compact and closed in $(X, \mathcal{T})$. Prove that $(X', \mathcal{T}')$ is a compact space containing $(X, \mathcal{T})$ as a subspace. [Note: then $(X', \mathcal{T}')$ is called the Alexandroff one-point compactification of $(X, \mathcal{T})$.]

Old Tests and Exams

Test 1, February 6, 2012

Please show all work. If a result is not in the text or in the exercises for Chapters 1-7, then it must be justified. If you are unsure, then justify the result.

1. Give an example of a nontrivial (neither discrete nor indiscrete) topology on the set $\mathbb{Z}$ of integers. Show why your example works.

2. Let $X = \{0, 1, 2\}$, $A = \{0, 1\}$, $B = \{1, 2\}$, and let $\mathcal{T} = \{\emptyset, A, B, X\}$. Show that $(X, \mathcal{T})$ is not a topological space.

3. Let (X, d) be a metric space, and define $\rho : X \times X \to \mathbb{R}$ by $\rho(x, y) = \min\{1, d(x, y)\}$. Prove that (X, ρ) is a metric space.

4. Let A be a subset of a metric space X. Prove that A is closed if and only if $\partial A \subseteq A$.

5. Let $\Delta : X \to X \times X$ be a map of metric spaces defined by $\Delta(x) = (x, x)$. Prove that Δ is continuous. (Let the metric on $X \times X$ be $d((x, y), (x', y')) = d_X(x, x') + d_X(y, y')$.

Test 2, March 19, 2012

Please show all work. If a result is not in the text or in the exercises for Chapters 1-12, then it must be justified. If you are unsure, then justify the result.

1. Suppose that A is a subset of a topological space X. Prove that the boundary ∂A is closed in X.

2. Prove that any subspace of a Hausdorff space is Hausdorff.

3. Prove that a real-valued function $f : X \to \mathbb{R}$ on a space X is continuous if for any $x \in \mathbb{R}$, the sets $f^{-1}((x, \infty))$ and $f^{-1}((-\infty, x))$ are both open in X.
   (**Hint: note that** $(a, b) = (-\infty, b) \cap (a, \infty)$).

4. Suppose that A and B are connected subsets of a space X such that $A \cap \overline{B} \neq \emptyset$. Prove that $A \cup B$ is connected.

5. Suppose that $(A, \mathcal{T}_A)$ is a subspace of a space $(X, \mathcal{T})$, and let $W \subseteq A$.

(a) Prove that if W is closed in A and A is closed in X, then W is closed in X.

(b) Give an example of a proper subset $W \subset A$ where W is closed in A but W is not closed in X.

Final Exam, April 12, 2012

- This exam is open book, closed notes.
- Please show all work. If a result is in the text or in the exercises for Chapters 1-13,15-16 please reference it (unless it is the question you are asked to prove). Otherwise, justify the result.
- All questions are equally weighted.

Part A: Do all six (6) questions

1. Prove that if A and B are bounded subsets of a metric space and $A \cap B \neq \emptyset$ then $\text{diam}(A \cup B) \leq \text{diam}A + \text{diam}B$.

2. Let $f : X \to Y$ be a map of topological spaces, where $X = (\mathbb{R}, \mathcal{T}_{\text{discrete}})$ and $Y = (\mathbb{R}, \mathcal{T}_{d_1})$. Define $f(x) = x$ for all $x \in X$. Prove that f is continuous, injective, and surjective, but f^{-1} is not continuous. Explain why this shows f is not a quotient map.

3. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = nx(1 - x^2)^{n^2}$, for $n \in \mathbb{N}$. Does (f_n) converge uniformly? Why or why not?

4. Let $X = A \cup B$, where A and B are closed subsets of X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, prove that

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is continuous on X.

5. A map $f : X \to Y$ is said to be an **open map** if for every open set U of X, the set $f(U)$ is open in Y. Prove that the projection map $\pi_1 : X \times Y \to X$ is an open map.

6. Let A be a subset of a topological space X. A point $x \in X$ is a limit point of A if for every open set U containing x, $(U \setminus \{x\}) \cap A \neq \emptyset$. Now let X be a Hausdorff space, A a subset of X, and x a limit point of A. Prove that every open set containing x contains infinitely many elements of A.

Part B: Do three (3) of the following five (5) questions

7. Let $f : X \to Y$ be a map of metric spaces. Prove that f is continuous if and only if whenever (x_n) is a sequence in X converging to a point $x \in X$ we have $(f(x_n))$ converges to $f(x)$ in Y.

8. Let A be a subset of a topological space X. Prove that A is dense in X if and only if the interior of $X \setminus A$ is empty.

9. Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected and if Y is connected, then X is connected.

10. Let X be a compact space. Prove that if A is a subset of X with no limit points, then A is finite.

11. (a) If $\{\mathcal{T}_i\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_i$ is a topology on X. Is $\bigcup \mathcal{T}_i$ a topology on X?

    (b) Let $\{\mathcal{T}_i\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all $\mathcal{T}_i$, and a unique largest topology which is contained in all of the $\mathcal{T}_i$.

    (c) If $X = \{a, b, c\}$, let $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$. Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.