Math 431: Metric Spaces and Topology
Solutions

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Assignment 1: Due January 27, 2017

2.3: Suppose that V, X, Y are sets with $V \subseteq X \subseteq Y$ and suppose that U is a subset of Y such that $X \setminus V = X \cap U$. Prove that

$$V = X \cap (Y \setminus U).$$

Solution: Let $x \in V$. Then $x \in X, Y$ since V is a subset of each, and $x \notin U$ since $x \notin X \setminus V$. Therefore $x \in X$ and $x \in Y \setminus U$, so $x \in X \cap (Y \setminus U)$ giving $V \subseteq X \cap (Y \setminus U)$.

Let $x \in X \cap (Y \setminus U)$, then $x \in X$ and $x \notin U$, so $x \notin X \cap U = X \setminus V$. Thus $x \in V$, so $X \cap (Y \setminus U) \subseteq V$.

Thus we have equality. ■

2.6: Suppose that for some set X and some indexing sets I, J we have $U = \bigcup_{i \in I} B_{i1}$ and $V = \bigcup_{j \in J} B_{j2}$ where each B_{i1}, B_{j2} is a subset of X. Prove that

$$U \cap V = \bigcup_{(i,j) \in I \times J} B_{i1} \cap B_{j2}.$$

Solution:

$$\begin{aligned} x \in U \cap V &\iff x \in U \text{ and } x \in V \\ &\iff x \in B_{i1} \text{ for some } i \in I, \text{ and } x \in B_{j2} \text{ for some } j \in J \\ &\iff x \in B_{i1} \cap B_{j2} \text{ for some } (i,j) \in I \times J \\ &\iff x \in \bigcup_{(i,j) \in I \times J} B_{i1} \cap B_{j2} \end{aligned}$$

■

3.4: Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x, 2x)$. Describe the following sets:

- $f([0,1])$
- $f^{-1}([0,1] \times [0,1])$
- $f^{-1}(D)$ where $D = \{(x,y) | x^2 + y^2 \leq 1\}$.

Solution:

- $f([0,1]) = \{(t, 2t) | 0 \leq t \leq 1\} =$ the line segment joining $(0,0)$ to $(1,2)$

- $f^{-1}([0,1]\times[0,1]) = [0,.5]$
- $f^{-1}(D)$ where $D = \{(x,y)|x^2+y^2 \le 1\}$. So $f^{-1}(D) = [-1/\sqrt{5}, 1/\sqrt{5}]$.

■

3.9: Let $f : X \to Y$ be a map and $A \subseteq X, C \subseteq Y$. Prove that

(a) $f(A) \cap C = f(A \cap f^{-1}(C))$

Solution:

$$\begin{aligned} y \in f(A)\cap C &\iff y \in f(A) \text{ and } y \in C \\ &\iff y = f(a) \text{ for some } a \in A, \text{ and } y \in C \\ &\iff a \in f^{-1}(C) \text{ and } a \in A \text{ and } y = f(a) \\ &\iff a \in A \cap f^{-1}(C) \text{ and } y = f(a) \\ &\iff y \in f(A \cap f^{-1}(C)) \end{aligned}$$

■

(b) If also $B \subseteq X$ and $f^{-1}(f(B)) = B$ then $f(A) \cap f(B) = f(A \cap B)$.

Solution: Let $C = f(B)$ in part (a), and we have the result. ■

4.3: Formulate and prove analogues of Exercises 4.1 and 4.2 for inf.

Solution:

- For 4.1: If $\emptyset \ne A \subseteq B \subseteq \mathbb{R}$ and B is bounded below, then A is bounded below and $\inf A \ge \inf B$.
  Proof: Since B is bounded below, $\exists k \in \mathbb{R}$ such that $k \le x$, $\forall x \in B$, and since $A \subseteq B$, $k \le x$, $\forall x \in A$. Thus A is bounded below. Next, assume $\inf A < \inf B$. Then $\exists a \in A$ such that $a < \inf B$. Since $a \in A \subseteq B$, this means $a \ge \inf B$. This is a contradiction, therefore $\inf A \ge \inf B$.
- For 4.2: If A, B are non-empty subsets of $\mathbb{R}$ which are bounded below, then $A \cup B$ is bounded below and

  $$\inf A \cup B = \min\{\inf A, \inf B\}$$

  Proof: Let A be bounded below by a and B be bounded below by b. Let $k = \min\{a, b\}$. Then $k \le a \le x$, $\forall x \in A$ and $k \le b \le y$, $\forall y \in B$, so $k \le z$, $\forall z \in A \cup B$. The second part can be shown by letting $a = \inf A$ and $b = \inf B$, and if $\inf A \cup B$ is greater than that minimum, either the $\inf A$ or $\inf B$ is violated since it would be a lower bound on both A and B.

■

4.5: Show that there is no rational number q such that $q^2 = 2$.

Solution: Assume $q^2 = 2$ where $q = \frac{m}{n}$ where m and n are natural numbers with no common factors. Then $\frac{m^2}{n^2} = 2$, or $m^2 = 2n^2$. Since m, n are relatively prime, n does not divide, unless $n = 1$. We can rule out this case by showing $1^2 = 1 < 2$ and $m^2 \geq 4$ for all $m \geq 2$. Thus 2 divides m^2, and this implies 2 divides m. Therefore $m = 2k$ for some $k \in \mathbb{N}$. Put this in above and we have $4k^2 = 2n^2$ or $2k^2 = n^2$. Now either $k = 1$ (rule out as before), k divides n (this would mean m, n have a common factor of k), or 2 divides n (which means m, n have a common factor of 2). None of these can happen. This is a contradiction. Therefore, there is no rational number whose square is 2. ■

4.8: Prove that between any two distinct real numbers there is an irrational number.

Solution: Let x and y be distinct real numbers with $x < y$. Then $x + \sqrt{2}$ and $y + \sqrt{2}$ are distinct real numbers. By Corollary 4.7, there exists a rational number, r, such that $x + \sqrt{2} < r < y + \sqrt{2}$. Then $x < r - \sqrt{2} < y$. From Exercise 4.5 above, we know that $\sqrt{2}$ is not rational, and thus $r - \sqrt{2}$ is also not rational. ■

4.11: Given a set of r non-negative real numbers $\{a_1, \ldots, a_r\}$, let $a = \max a_i$. Prove that for any positive integer n,

$$a^n \leq a_1^n + \cdots + a_r^n \leq r a^n.$$

By taking n^{th} roots throughout, deduce that

$$a \leq (a_1^n + \cdots + a_r^n)^{1/n} \leq r^{1/n} a,$$

and hence that

$$\lim_{n \to \infty} (a_1^n + \cdots + a_r^n)^{1/n} = a.$$

Solution: Since $a_i \geq 0$ and $a = \max a_i$, we have $0 \leq a_i \leq a$ for $i = 1, \ldots, r$, and $a = a_k$ for some k. This means $0 \leq a_i^n \leq a^n$ as well, since x^n is an increasing function for non-negative x. Thus

$$a^n = a_k^n \leq a_1^n + \cdots + a_k^n + \cdots + a_r^n \leq a^n + a^n + \cdots + a^n = r a^n$$

Since $x^{1/n}$ is a strictly increasing function, taking the n^{th} root preserves the inequalities. For the last statement, we merely need that $\lim_{n \to \infty} r^{1/n} = 1$ for positive integers r (this is a well-known fact, though it would be good to prove it) and the Squeeze Theorem for sequences. ■

Assignment 2: Due February 10, 2017

1. For $x = (-2, 1)$ and $y = (3, 4)$, compute the distance from x to y in each of the following metrics:

   (a) The discrete metric.
   Solution: Since $x \neq y$, the distance using the discrete metric is 1. ■

   (b) d_1, d_2, and d_∞ as described in Example 5.7
   Solution:

$$\begin{aligned} d_1(x, y) &= |-2-3| + |1-4| = 5 + 3 = 8 \\ d_2(x, y) &= \sqrt{(-2-3)^2 + (1-4)^2} = \sqrt{25+9} = \sqrt{34} \\ d_\infty(x, y) &= \max\{|-2-3|, |1-4|\} = \max\{5, 3\} = 5 \end{aligned}$$

■

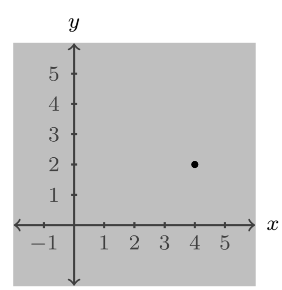
2. Describe pictorially (on a graph) the set of points $x \in \mathbb{R}^2$ whose distance from $(4, 2)$ is less than or equal to 1 with respect to the following metrics:

   (a) The discrete metric.

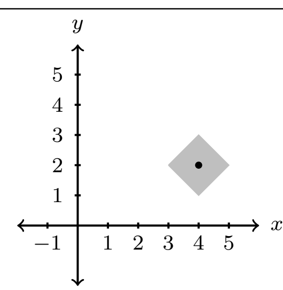
   (b) d_1, d_2, and d_∞ as described in Example 5.7

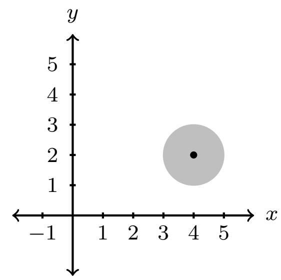
Solution:

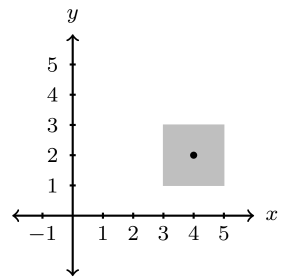
$B_1^{discrete}((4, 2)) = \mathbb{R}^2$:



$B_1^{d_1}((4, 2))$:



$B_1^{d_2}((4,2))$: 

$B_1^{d_\infty}((4,2))$: 

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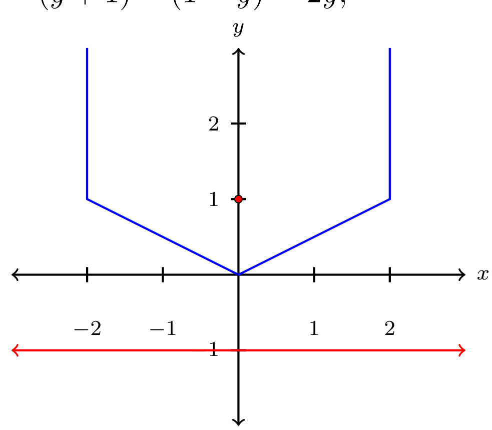
3. From geometry, a parabola is the collection of points which are equal distance from a fixed point and a given line. Sketch, with justification, the parabola defined by the point $(0,1)$ and the line $y = -1$ using the metric d_1 defined in Example 5.7.

   Solution: The parabola determined by the given point and line is the graph in blue. First, the equation which generates this graph is $|x| + |y-1| = |y+1|$, as can be seen from equating the two distances. In other words, $|x| = |y+1| - |y-1|$. Now clearly, $y+1 > y-1$. If $y - 1 \geq 0$ then so is $y+1$, and so $|x| = (y+1) - (y-1) = 2$. thus for $y \geq 1$, we have $|x| = 2$. This gives the two vertical lines in the graph.

   Now if $y < 0$, we see that $|y-1| \geq |y+1|$. This would mean for $y < 0$ we have $|x| < 0$, which cannot happen.

   This leaves $0 \leq y < 1$. In the case $|y-1| = 1-y$ and so $|x| = (y+1) - (1-y) = 2y$,



   which gives the two slanted lines from the origin, in the graph.

■

4. Prove that $d(x,y) = |e^x - e^y|$ defines a metric on $\mathbb{R}$. Describe the set of points in $\mathbb{R}$ whose distance from 1 is at most 5 under this metric.

Solution:

M1: Since $d(x,y) = |e^x - e^y|$ we always have the value non-negative. If $d(x,y) = 0$, then we must have $e^x = e^y$, and since the exponential function is injective, $x = y$.

M2: $d(x,y) = |e^x - e^y| = |e^y - e^x| = d(y,x)$.

M3: $d(x,z) = |e^x - e^z| = |e^x - e^y + e^y - e^z| \leq |e^x - e^y| + |e^y - e^z| = d(x,y) + d(y,z)$ using the triangle inequality for absolute values.

Find x such that $d(x,1) \leq 5$, so $|e^x - e| \leq 5$ which means $-5 + e \leq e^x \leq 5 + e$. Since $-5 + e < 0$, there will be no lower bound on x, but there will be an upper bound of $\ln(5+e)$. So, $x \in (-\infty, \ln(5+e)]$. ■

5. Let X be a metric space with metric d. Show that $D : X \times X \to \mathbb{R}$ defined by

$$D(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is also a metric on X. Also show that X is a bounded set in the metric space (X, D).

Solution:

M1: Since $d(x,y) \geq 0$, and $1 + d(x,y) \geq 1$, we must have $D(x,y) \geq 0$. Also, $D(x,y) = 0$ only when $d(x,y) = 0$ which means $x = y$ since d is a metric.

M2: Since $d(x,y) = d(y,x)$, we also have symmetry for D.

M3: First, we will show that $f(x) = \dfrac{x}{1+x}$ is increasing for all $x \neq -1$. The first derivative, $f'(x) = \dfrac{1}{(1+x)^2}$ is strictly positive where it is defined, and so f is increasing on its domain.

Now,

$$\begin{aligned}
D(x,z) &= \frac{d(x,z)}{1+d(x,z)} \\
&\le \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \qquad \text{since } d \text{ is a metric} \\
&= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)} \\
&\le \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \\
&\qquad \text{since the denominator is smaller in each fraction} \\
&= D(x,y) + D(y,z)
\end{aligned}$$

We have show that the function $f(x) = \dfrac{x}{1+x}$ is always increasing, and it is clear, after applying l'Hôpital's Rule, that $|f(x)| \le 1$. This means $D(x,y) \le 1$ for all $x, y \in X$, and so X is bounded. ■

5.2: Given points x, y, z, t in a metric space (X, d), prove that

$$|d(x,y) - d(z,t)| \le d(x,z) + d(y,t)$$

Solution:

$$\begin{aligned}
|d(x,y) - d(z,t)| &= |d(x,y) - d(y,z) + d(y,z) - d(z,t)| \\
&\le |d(x,y) - d(y,z)| + |d(y,z) - d(z,t)| \\
&\le d(x,z) + d(y,t) \qquad \text{Using exercise 5.1}
\end{aligned}$$

■

5.5: Suppose that x, y are distinct points in a metric space (X, d) and let $\epsilon = d(x,y)/2$. Prove that $B_\epsilon(x)$ and $B_\epsilon(y)$ are disjoint.

Solution: Assume that $B_\epsilon(x)$ and $B_\epsilon(y)$ are not disjoint. Then there is $z \in B_\epsilon(x) \cap B_\epsilon(y)$. Thus

$$d(x,y) \le d(x,z) + d(z,y) < \epsilon + \epsilon = 2\epsilon = d(x,y).$$

However, $d(x,y) \not< d(x,y)$. Therefore there cannot be an element in the intersection, and so the sets are disjoint. ■

5.18: Suppose that in a metric space X we have $B_s(x) = B_r(y)$ for some $x, y \in X$ and some positive real numbers r, s. Is $x = y$? Is $r = s$?

Solution: Let $X = \mathbb{R}$ with the discrete metric. Then $B_2(5) = \mathbb{R} = B_\pi(\sqrt{2})$. Thus it is not required that $r = s$ or that $x = y$. ∎

6. Let (X, d) be a metric space, and let P be the set of all non-empty subsets of X, that is $P = \{A | A \subseteq X, A \neq \emptyset\}$. We can define the distance between two elements of P by $D(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$. Then (P, D) is not a metric space. In what ways does it fail? Give an example of each type of failure.

   Solution:

   M1: Since d is a metric, $d(a, b) \geq 0$, and so $D(A, B) = \inf \{d(a, b) | a \in A, b \in B\} \geq 0$. However, $D(A, B) = 0$ does not imply $A = B$. Consider $A = [-1, 0]$ and $B = [0, 1]$ in the usual metric for $\mathbb{R}$. $A \neq B$ but $D(A, B) = 0$.

   M2: $D(A, B) = \inf \{d(a, b) | a \in A, b \in B\} = \inf \{d(b, a) | a \in A, b \in B\} = D(B, A)$.

   M3: The triangle inequality does not always hold. For example, take A and B as above, and let $C = [1, 2]$. Then $D(A, C) = 1 > D(A, B) + D(B, C) = 0 + 0$.

   ∎

7. Let $A = \{(x, y) \in \mathbb{R}^2 | 0 \leq x, y\}$. Determine whether A is open, closed, or neither with respect to: the discrete metric, d_1, d_2 and d_∞.

   Solution: Let $B = X \setminus A = \{(x, y) \in \mathbb{R}^2 | x < 0 \text{ or } y < 0\}$. In each case we need to consider whether A and/or B is open with respect to the metric.

   For the discrete metric, we know from examples 5.35 and 6.2 that any set in a discrete metric space is **both** open and closed.

   For metrics d_1, d_2, d_∞ we will show that A is closed. Let $(x, y) \in B$, then at least one of x or y is negative. Without loss of generality, assume it is x. Claim that $B_{-\frac{x}{2}}((x, y)) \subseteq B$. We will show that for $(a, b) \in B_{-\frac{x}{2}}((x, y))$ we must have $a < 0$, and thus $(a, b) \in B$.

$$|a - x| = \sqrt{(a - x)^2} \leq d((a, b), (x, y)) < -\frac{x}{2}$$
$$\frac{3x}{2} < a < \frac{x}{2} < 0$$

∎

5.7: Show that if S is a bounded set in $(\mathbb{R}^n, d_2)$, then S is contained in

$$[a, b] \times [a, b] \times \cdots \times [a, b]$$

for some $a, b \in \mathbb{R}$.

Solution: Since S is bounded, there exists a point $x \in \mathbb{R}^n$ and a non-negative number k such that $d(s, x) < k$ for all $s \in S$. Then $S \subseteq B_k(x)$. Thus for $y \in B_k(x)$ we have, for each $i = 1, \ldots, n$, $|y_i - x_i| \leq d_2(y, x) < k$. So $-k + x_i < y_i < k + x_i$. Pick $b = \max\{k + x_i\}$ and $a = \min\{-k + x_i\}$. Then $S \subseteq B_k(x) \subseteq [a, b] \times [a, b] \times \cdots \times [a, b]$. ■

5.14: Show that for any $x, y \in \mathbb{R}^n$,

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq n d_\infty(x, y).$$

Solution:

$$\begin{aligned}
\mathbf{d}_\infty(\mathbf{x}, \mathbf{y}) &= \max\{|x_i - y_i|\} \\
&= \max\left\{\sqrt{(x_i - y_i)^2}\right\} \\
&\leq \mathbf{d_2}(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} && \text{since } \sqrt{a^2} \leq \sqrt{a^2 + b^2} \\
&\leq \sqrt{(x_1 - y_1)^2} + \cdots + \sqrt{(x_n - y_n)^2} && \text{by the triangle inequality.} \\
&= \mathbf{d_1}(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots |x_n - y_n| \\
&\leq \max\{|x_i - y_i|\} + \cdots + \max\{|x_i - y_i|\} \\
&= n \max\{|x_i - y_i|\} = \mathbf{n}\mathbf{d}_\infty(\mathbf{x}, \mathbf{y})
\end{aligned}$$

■

6.3: Prove that any finite subset of a metric space X is closed in X.

Solution: Let $F = \{f_1, \ldots, f_n\}$ be finite, and let $x \in X \backslash F$. Define $\epsilon = \frac{1}{2}\min\{d(x, f_i)\}$ and let $y \in B_\epsilon(x)$. Now $y \neq f_i$, since $d(x, y) < \epsilon < 2\epsilon \leq d(x, f_i)$ for any i. Thus $y \in X \setminus F$ and so $B_\epsilon(x) \subseteq X \setminus F$ which means $X \setminus F$ is open and so F is closed. ■

6.16: For a point x and a non-empty subset A of a metric space (X, d), define $d(x, A) = \inf\{d(x, a) | a \in A\}$.

(a) Prove that $d(x, A) = 0$ iff $x \in \overline{A}$.
Solution: If $d(x, A) = 0$ then either $x \in A \subseteq \overline{A}$ or for every $\epsilon > 0$ we have $B_\epsilon(X) \cap A \neq \emptyset$ which means x is a limit point of A, and thus is in $\overline{A}$.
If $x \in \overline{A}$ then $x \in A$ and so $d(x, A) = 0$, or for every $\epsilon > 0$, we have $B_\epsilon(x) \cap A \neq \emptyset$ and there is a $a_\epsilon \in A$ such that $d(x, a_\epsilon) < \epsilon$, so $d(x, A) = \inf\{d(x, a) | a \in A\} < \epsilon$ for all $\epsilon > 0$ and thus is 0. ■

(b) Show that if y is another point in X, then $d(x, A) \leq d(x, y) + d(y, A)$.
Solution: Since $d(x, A) = \inf\{d(x, a)|a \in A\}$ we have $d(x, A) \leq d(x, a)$ for all $a \in A$. Using the triangle inequality, we have $d(x, a) \leq d(x, y) + d(y, a)$. Also $\inf\{K + a_i|i \in I\} = K + \inf\{a_i|i \in I\}$ for any constant K. Therefore $\inf\{d(x, a)\} \leq d(x, y) + \inf\{d(y, a)\}$, which is what we are trying to show. ■

(c) Prove that $x \mapsto d(x, A)$ gives a continuous map from X to $\mathbb{R}$.
Solution: Let $\epsilon > 0$ and pick $\delta = \epsilon$. Now $d(x, A) \leq d(x, y) + d(y, A)$, so $d(x, A) - d(y, A) \leq d(x, y)$. Similarly, we have $d(y, A) - d(x, A) \leq d(x, y)$, thus $-d(x, y) \leq d(x, A) - d(y, A) \leq d(x, y)$.
So, if $d(x, y) < \delta$, then $|d(x, A) - d(y, A)| \leq d(x, y) < \delta = \epsilon$. ■

Assignment 3: Due February 24, 2017

6.18: Prove that a finite subset of a metric space has no limit points.
Solution: Assume that $F = \{f_1, \dots, f_n\}$ has a limit point f. Let

$$\epsilon = \frac{1}{2}\min\{d(f, f_i) | i = 1, \dots, n\}.$$

Then $B_\epsilon(f) \cap B_\epsilon(f_i) = \emptyset$ for all i. Therefore f cannot be a limit point. ■

6.23: For a subset A of a metric space X, prove:

(a) $A^\circ = A \setminus \partial A = \overline{A} \setminus \partial A$,
Solution: First we observe that for subsets A, B, C of X, that $x \in A \setminus (B \setminus C)$ if and only if $x \in A$ and $x \notin B \setminus C$ if and only if ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \in B$ and $x \in C$) if and only if $x \in (A \setminus B) \cup (A \cap B \cap C)$.
Then $A \setminus \partial A = A \setminus (\overline{A} \setminus A^\circ) = (A \setminus \overline{A}) \cup (A \cap \overline{A} \cap A^\circ) = \emptyset \cup A^\circ = A^\circ$. So the first equality is shown.
Similarly, $\overline{A} \setminus \partial A = \overline{A} \setminus (\overline{A} \setminus A^\circ) = (\overline{A} \setminus \overline{A}) \cup (\overline{A} \cap \overline{A} \cap A^\circ) = \emptyset \cup A^\circ = A^\circ$. Therefore we have the equalities indicated. ■

(b) $\overline{X \setminus A} = X \setminus A^\circ$,
Solution: $x \in X \setminus A^\circ$ if and only if $x \notin A^\circ$ if and only if for every $\epsilon > 0$ we have $B_\epsilon(x) \cap (X \setminus A) \neq \emptyset$ if and only if $x \in \overline{X \setminus A}$. ■

(c) $\partial A = \overline{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$,
Solution: First, $x \in \partial A = \overline{A} \setminus A^\circ$ if and only if $x \in \overline{A}$ and $x \notin A^\circ$ if and only if $x \in \overline{A}$ and for every $\epsilon > 0$, $B_\epsilon(x) \cap (X \setminus A) \neq \emptyset$ if and only if $x \in \overline{A} \cap \overline{X \setminus A}$.
Next, $\partial(X \setminus A) = \overline{X \setminus A} \cap \overline{X \setminus (X \setminus A)} = \overline{X \setminus A} \cap \overline{A} = \partial A$. ■

(d) ∂A is closed in X.
Solution: From above, ∂A is the intersection of two closed set and is therefore closed. ■

6.27: Prove that the metrics $d^{(2)}$, $d^{(3)}$ in Exercise 5.12 are topologically equivalent to d.
Solution: First, we will show that $d^{(2)}$ and $d^{(3)}$ are Lipschitz equivalent by proving

$$d^{(3)}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq \min\{1, d(x, y)\} = d^{(2)}(x, y) \leq 2d^{(3)}(x, y).$$

Now $d(x, y) < 1 + d(x, y)$, so $d^{(3)}(x, y) < 1$, and $d(x, y) \leq d(x, y) + d(x, y)^2$ so $d^{(3)}(x, y) \leq d(x, y)$. Thus $d^{(3)} \leq d^{(2)}$.

Next, $1 + d(x,y) \leq 2d(x,y)$ if $d(x,y) \geq 1$, so $1 \leq 2d^{(3)}$ if $d(x,y) \geq 1$. If $d(x,y) < 1$ then $d(x,y)^2 < d(x,y)$ and therefore $d(x,y)+d(x,y)^2 < 2d(x,y)$. Therefore $d^{(2)}(x,y) \leq 2d^{(3)}(x,y)$ and we therefore have $d^{(2)}$ and $d^{(3)}$ are Lipschitz equivalent.

Second, we will show that $d^{(2)}$ is topologically equivalent to d which will complete the prove (since topological equivalence is an equivalence relation).

Now if $\epsilon \leq 1$ then $B^d_\epsilon(x) = B^{d^{(2)}}_\epsilon(x)$. Let U be d-open. Then for every $x \in U$ there exists $1 > \epsilon > 0$ such that $B^d_\epsilon(x) = B^{d^{(2)}}_\epsilon(x) \subseteq U$. We can impose the upper bound since we have for $r < s$ $B_r(x) \subset B_s(x)$. This means U is $d^{(2)}$-open. The reverse is true for the same reason. ■

7.2: Give an example of two topologies $\mathcal{T}_1, \mathcal{T}_2$ on the same set such that neither contains the other.
Solution: Let $X = \{0,1\}$ and $\mathcal{T}_1 = \{\emptyset, \{0\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{1\}, X\}$. Clearly, $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \not\subseteq \mathcal{T}_1$. ■

7.3: Show that the intersection of two topologies on the same set X is also a topology on X, but that their union may or may not be a topology. Does this first result extend to the intersection of an arbitrary family of topologies on X?
Solution: We will show that the intersection of an arbitrary family of topologies on X is a topology on X.

Let T_i be topologies on X for $i \in I$ and let $T = \bigcap_{i\in I} T_i$.

T1: Since $\emptyset$ and X are in each T_i, they are also in T.

T2: Let $U, V \in T$. Then $U, V \in T_i$ for all $i \in I$. Since each T_i is a topology, $U \cap V \in T_i$, again for each $i \in I$. Thus $U \cap V \in T$.

T3: Let $U_j \in T$ for $j \in J$. Then $U_j \in T_i$ for all $j \in J$ and all $i \in I$ Thus $\bigcup_{j\in J} U_j \in T_i$ for all $i \in I$ and therefore $\bigcup_{j \in J} U_j \in T$.

To show that the union of two topologies does not have to be a topology, consider $X = \{0,1,2\}$, $T_1 = \{\emptyset, \{1\}, X\}$ and $T_2 = \{\emptyset, \{2\}, X\}$. Then $T = T_1 \cup T_2 = \{\emptyset, \{1\}, \{2\}, X\}$ which is not a topology since $\{1\} \cup \{2\} \notin T$. ■

7.4: Prove that we get a topology for $\mathbb{N} = \{1, 2, 3, \dots\}$ by taking the open sets to be $\emptyset, \mathbb{N}$ and $\{1, 2, 3, \dots, n\}$ for each $n \in \mathbb{N}$.
Solution: Let $T = \{\emptyset, \mathbb{N}, \{1\}, \{1,2\}, \dots\}$

T1: $\emptyset, \mathbb{N}$ are in T by definition of T.

T2: Let $U, V \in T$. Note: $\emptyset \cap U = \emptyset \in T$ and $\mathbb{N} \cap U = U \in T$. Let $U = \{1, \dots, n\}$ and $V = \{1, \dots, m\}$ and assume, without loss of generality, that $n < m$. Then $U \cap V = U \in T$.

T3: Let $U_i \in T$ where $U_i = \{1, \ldots, n_i\}$ or possibly $\emptyset$ or $\mathbb{N}$. If the empty set is one of the U_i it can be ignored, and if $\mathbb{N}$ is a member of the collection, then the union is $\mathbb{N}$. Let $W = \{n_i\}_{i \in I}$. If W is not bounded above, then $\bigcup_{i\in I} U_i = \mathbb{N}$. Otherwise, there is a largest value n_k, and $\bigcup_{i\in I} U_i = U_k$.

■

7.6: Let $\mathcal{T}$ be the collection of all subsets of $\mathbb{R}$ consisting of $\emptyset, \mathbb{R}$ together with all intervals of the form $(-\infty, b)$. Show that $\mathcal{T}$ is a topology for $\mathbb{R}$.
Solution:

T1: $\emptyset, \mathbb{R}$ are in $\mathcal{T}$ by definition of $\mathcal{T}$.

T2: Let $U, V \in T$. Note: $\emptyset \cap U = \emptyset \in T$ and $\mathbb{R} \cap U = U \in T$. Let $U = (-\infty, n)$ and $V = (-\infty, m)$ and assume, without loss of generality, that $n < m$. Then $U \cap V = U \in T$.

T3: Let $U_i \in T$ where $U_i = (-\infty, n_i)$ or possibly $\emptyset$ or $\mathbb{R}$. If the empty set is one of the U_i it can be ignored, and if $\mathbb{R}$ is a member of the collection, then the union is $\mathbb{R}$. Let $W = \{n_i\}_{i\in I}$. If W is not bounded above, then $\bigcup_{i\in I} U_i = \mathbb{R}$. Otherwise, let $w = \sup W$. Clearly, $\bigcup_{i\in I} U_i \subseteq (-\infty, w)$ since each $U_i \subseteq (-\infty, w)$. Let $x \in (-\infty, w)$. Then $x < w$. If $x \notin U_i$ for all $i \in I$ then $n_i < x$ for all $i \in I$ and hence x is an upper bound for W, which cannot happen since w is the least upper bound for W. Thus $x \in U_i$ for some $i \in I$ and is thus in the union. This gives us equality, and so $\bigcup_{i\in I} U_i \in \mathcal{T}$.

■

8.1: Let $f : X \to Y$ be a map of topological spaces. Prove that f is continuous in the following cases:

(a) $X = Y$ and f is the identity map.

(b) f is a constant map.

(c) $\mathcal{T}_X$ is discrete.

(d) $\mathcal{T}_Y$ is indiscrete.

Solution:[1]

(a) Let $U \in \mathcal{T}$, then $f^{-1}(U) = U \in \mathcal{T}$. So f is continuous.

(b) Let $y_0 \in Y$ be a constant such that $f(x) = y_0 \quad \forall x \in X$. Let $U \in \mathcal{T}_Y$. If $y_0 \in U$, then $f^{-1}(U) = f^{-1}(y_0) = X \in \mathcal{T}_X$, otherwise $f^{-1}(U) = \emptyset \in \mathcal{T}_X$. So f is continuous.

[1]This solution is by Ms. Huntemann. The notations in red are my corrections. I have reformatted some of the solutions for legibility in the LaTeX code.

(c) Let $U \in \mathcal{T}_Y$, then $f^{-1}(U) = V \subseteq X$ and since $\mathcal{T}_X$ is the discrete topology, $V \in \mathcal{T}_X$. So f is continuous.

(d) Since $\mathcal{T}_Y = \{\emptyset, Y\}$, we only have to test those two sets. $f^{-1}(\emptyset) = \emptyset \in \mathcal{T}_X$ and $f^{-1}(Y) = X \in \mathcal{T}_X$. So f is continuous.

■

8.5: Prove that the set of all open intervals $\{(a,b) : a, b \in \mathbb{R}, a < b\}$ is a basis for the usual topology on $\mathbb{R}$.

Solution:[2] Let $(\mathbb{R}, d)$ be a metric space with the usual metric. Then the only open sets in the metric space are of the form $\bigcup_{i \in I}(a_i, b_i)$ with $a_i < b_i$ and $a_i, b_i \in \mathbb{R}$. (See Exercise 5.13 which we proved in class.) These are also the sets in the topology. So all sets in the topology are either of the form (a,b) or are the arbitrary union of sets of this form. Therefore $\{(a,b) : a, b \in \mathbb{R}, a < b\}$ is a basis. ■

9.5: Give either a proof of, or a counterexample to, each of the following:

(a) If $f : X \to Y$ is a continuous map of topological spaces and $A \subseteq X$ is closed, then $f(A) \subseteq Y$ is closed.

Solution:[3] This is not true. A counterexample would be the following: Let $Y = \{0, 1\}$ and $\mathcal{T}_Y = \{\emptyset, \{0\}, Y\}$, and $X = \mathbb{R}$ and $\mathcal{T}_X = \{\emptyset, (-\infty, 0), [0, \infty), X\}$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases} \tag{0.0.1}$$

In this case, if $A \subseteq X$ is closed, then $A \in \mathcal{T}_X$. But $f([0,\infty)) = 0$ is not closed in Y since $Y \setminus \{0\} = \{1\}$ is not open in Y. ■

(b) If A is open in a topological space X and $B \subseteq X$ then $A \cap \overline{B} = \overline{A \cap B}$.

Solution:[4] This is not true. A counterexample would be the following: Let $X = \mathbb{R}$ with the usual topology, and let $A = (0, 2)$ and $B = [1, 3]$. Then A is open in X and $B \subseteq X$. But $A \cap \overline{B} = [1, 2) \neq [1, 2] = \overline{A \cap B}$. ■

(c) If $f : X \to Y$ is a continuous map of topological spaces and $B \subseteq Y$ then $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

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Solution:[5] This is not true. A counterexample would be the following: Let $X = Y = \mathbb{R}$ with the usual topology and let

$$f(x) = \begin{cases} 0 & \mathit{if}\ x < 0 \\ x & \mathit{if}\ 0 \leq x \leq 1 \\ 1 & \mathit{if}\ x > 1 \end{cases} \tag{0.0.2}$$

which is a continuous function. Then let $B = (0,1)$, so that $\overline{B} = [0,1]$, $f^{-1}(B) = (0,1)$, and $f^{-1}(\overline{B}) = \mathbb{R}$. Therefore $\overline{f^{-1}(B)} = [0,1] \neq f^{-1}(\overline{B}) = \mathbb{R}$. ■

9.7: Prove that a map $f : X \to Y$ of topological spaces is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$.
Solution:[6] First, let f be continuous and $A \subseteq X$. We know that $f(A) \subseteq \overline{f(A)}$, therefore $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$. Since f is continuous and $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed. Since $\overline{A}$ is the smallest closed set containing A, we have $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Also,

$$f(f^{-1}(\overline{f(A)})) = \overline{f(A)} \cap f(X) = \subseteq \text{ since there may be points in } Y \text{ not in } f(X). \overline{f(A)}.$$

Therefore $f(\overline{A}) \subseteq \overline{f(A)}$.

Now let $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. Let $V \subseteq Y$ be closed, then $f^{-1}(V) \subseteq X$, so $f(\overline{f^{-1}(V)}) \subseteq \overline{f(f^{-1}(V))} \subseteq \overline{V} = V$. Therefore, $f^{-1}(f(\overline{f^{-1}(V)})) \subseteq f^{-1}(\overline{f(f^{-1}(V))}) \subseteq f^{-1}(V) \subseteq \overline{f^{-1}(V)}$. We also know that $\overline{f^{-1}(V)} \subseteq f^{-1}(f(\overline{f^{-1}(V)}))$, therefore $f^{-1}(V) = \overline{f^{-1}(V)}$ is closed and we have that f is continuous. ■

9.11: Let $A_1, \ldots, A_m$ be subsets of a topological space X. Prove that the interior of $\bigcap_{i=1}^m A_i$ equals $\bigcap_{i=1}^m \mathring{A}_i$.
Solution:[7] Let $x \in \bigcap_{i=1}^m \mathring{A}_i$, then $x \in \mathring{A}_1, x \in \mathring{A}_2, \ldots, x \in \mathring{A}_m$. Then for each $i \in \{1, \ldots, m\}$, there exists a $U_i \in \mathcal{T}$ such that $x \in U_1 \subseteq A_1, \ldots, x \in U_m \subseteq A_m$. Then $x \in \bigcap_{i=1}^m U_i \subseteq \bigcap_{i=1}^m A_i$ and since all $U_i \in \mathcal{T}$, we also have $\bigcap_{i=1}^m U_i \in \mathcal{T}$. Therefore $x \in \overset{\circ}{\bigcap_{i=1}^m} A_i$ and then $\bigcap_{i=1}^m \mathring{A}_i \subseteq \overset{\circ}{\bigcap_{i=1}^m} A_i$.

Let $x \in \overset{\circ}{\bigcap_{i=1}^m} A_i$. Then there exists a $U \in \mathcal{T}$ such that $x \in U \subseteq \bigcap_{i=1}^m A_i$. Therefore, $x \in U \subseteq A_1, x \in U \subseteq A_2, \ldots, x \in U \subseteq A_m$. So $x \in \mathring{A}_1, \ldots, x \in \mathring{A}_m$. Therefore, $x \in \bigcap_{i=1}^m \mathring{A}_i$, and $\overset{\circ}{\bigcap_{i=1}^m} A_i \subseteq \bigcap_{i=1}^m \mathring{A}_i$. ■

9.15: Let A be a subspace of a topological space X. Show that $\overline{A}$ is the disjoint union of $\mathring{A}$ and ∂A. Deduce that if B is another subspace of X such that $B \cap A \neq \emptyset$ then either

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$B \cap \partial A \neq \emptyset$ or $B \cap \mathring{A} \neq \emptyset$.
Solution:[8] We know that $\mathring{A}$ and ∂A are disjoint since $\partial A = \overline{A} \setminus \mathring{A}$.

Let $a \in \overline{A}$, then either $a \in \mathring{A}$ or $a \in \overline{A} \setminus \mathring{A}$. Therefore $\overline{A} \subseteq \mathring{A} \cup \partial A$.

Let $a \in \mathring{A} \cup \partial A$, then either $a \in \mathring{A} \subseteq \overline{A}$ or $a \in \partial A$ (the or is strict since $\mathring{A}$ and ∂A are disjoint). If $a \in \partial A$, we know $a \notin \mathring{A}$, and since $\partial A = \overline{A} \setminus \mathring{A}$, we have $a \in \overline{A}$. Therefore $\mathring{A} \cup \partial A \subseteq \overline{A}$.

Therefore, $\overline{A} = \partial A \cup \mathring{A}$.

Now let $B \subseteq X$ with $B \cap A \neq \emptyset$. Let $x \in B \cap A$, then $x \in B$ and $x \in A \subseteq \overline{A} = \mathring{A} \cup \partial A$. Thus $x \in B$ and either $x \in \mathring{A}$ or $x \in \partial A$, so either $x \in B \cap \mathring{A}$ or $x \in B \cap \partial A$. Therefore either $B \cap \mathring{A} \neq \emptyset$ or $B \cap \partial A \neq \emptyset$. ■

10.5: Suppose that $(A, \mathcal{T}_A)$ is a subspace of a space $(X, \mathcal{T})$ and that $V \subseteq X$ is closed in X. Prove that $V \cap A$ is closed in $(A, \mathcal{T}_A)$.
Solution:[9] Since V is closed, we know that $X \setminus V \in \mathcal{T}$ and by definition of the subspace topology $A \cap (X \setminus V) \in \mathcal{T}_A$. We also know that $A \cap (X \setminus V) = A \setminus (V \cap A)$, so $A \setminus (V \cap A) \in \mathcal{T}_A$. Therefore $V \cap A$ is closed in A. ■

10.12: Suppose that $\mathcal{S}$ is the Seirpinski space of Example 7.7. Find the product topology $\mathcal{S} \times \mathcal{S}$.
Solution: $\mathcal{S} = \{\emptyset, \{1\}, \{0,1\}\}$.

$$\begin{aligned} \mathcal{S} \times \mathcal{S} = &\{\emptyset, \{(0,0),(0,1),(1,0),(1,1)\}, \\ &\{(1,0),(1,1)\}, \{(0,1),(1,1)\}, \\ &\{(1,1)\}, \{(1,1),(0,1),(1,0)\}\} \end{aligned}$$

■

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Assignment 4: Due March 17, 2017

10.15: (a) Prove that if W is open in a topological product $X \times Y$ then $p_X(W)$ is open in X and $p_Y(W)$ is open in Y.

Solution: If W is open in the topological product $X \times Y$, then $W = \bigcup_{i \in I} U_i \times V_i$ for $U_i \in \mathcal{T}_X$ and $V_i \in \mathcal{T}_Y$ for all $i \in I$. Then

$$p_X(W) = p_X\left(\bigcup_{i \in I} U_i \times V_i\right) = \bigcup_{i \in I} p_X(U_i \times V_i) = \bigcup_{i \in I} U_i \in \mathcal{T}_X$$

$$p_Y(W) = p_Y\left(\bigcup_{i \in I} U_i \times V_i\right) = \bigcup_{i \in I} p_Y(U_i \times V_i) = \bigcup_{i \in I} V_i \in \mathcal{T}_Y$$

■

(b) Give an example of a closed set $W \subset \mathbb{R} \times \mathbb{R}$ whose projection $p_1(W)$ on the x-axis is not closed in $\mathbb{R}$.

Solution: Let $W = \{(x, 1/x) | x > 0\}$. This set is closed. To see this, take $(a, b) \notin W$, and let $\epsilon = \frac{1}{2} d((a, b), W)$ where this distance is defined in Exercise 6.16. Then $B_\epsilon((a, b)) \cap W = \emptyset$, showing that $\mathbb{R}^2 \setminus W$ is open, thus W is closed.

Now $p_X(W) = (0, \infty)$ which is open but not closed. ■

10.16: Suppose that X, Y are spaces and that $A \subseteq X, B \subseteq Y$. Prove that

(a) the interior of $A \times B$ is $\mathring{A} \times \mathring{B}$.

Solution: Clearly $\mathring{A} \times \mathring{B}$ is an open subset of $A \times B$ and is thus contained in the interior.

Next, the interior of $A \times B$ is an open set in the product topology, and is thus equal to $\bigcup_{i \in I} U_i \times V_i$ for open sets $U_i \subseteq X$ and $V_i \subseteq Y$. Also, each $U_i \subseteq A$ and $V_i \subseteq B$. This means that each $U_i \subseteq \mathring{A}$ and $V_i \subseteq \mathring{B}$, and so $U_i \times V_i \subseteq \mathring{A} \times \mathring{B}$ for all $i \in I$. This gives us the desired equality. ■

(b) $\overline{A \times B} = \overline{A} \times \overline{B}$.

Solution: Let $(a, b) \in \overline{A \times B}$ and let $a \in W \in \mathcal{T}_X$. Then $W \times Y$ is open in the product topology and so $(W \times Y) \cap (A \times B) \neq \emptyset$. Then $p_X((W \times Y) \cap (A \times B)) = W \cap A \neq \emptyset$ so $a \in \overline{A}$. Similarly for $b \in \overline{B}$.

Next, let $(a, b) \in \overline{A} \times \overline{B}$ and let W be an open set containing (a, b). Then $p_X(W \cap (A \times B)) = p_X(W) \cap A \neq \emptyset$ since $a \in \overline{A} \cap p_X(W)$. Therefore $W \cap (A \times B) \neq \emptyset$ so $(a, b) \in \overline{A \times B}$. ■

(c) $\partial(A \times B) = ((\partial A) \times \overline{B}) \cup (\overline{A} \times (\partial B))$.
Solution: First we will show that $(U \times V) \setminus (A \times B) = ((U \setminus A) \times V) \cup (U \times (V \setminus B))$.

$$\begin{aligned}(U \times V) \setminus (A \times B) &= \{(x,y) \in U \times V | (x,y) \notin A \times B\} \\ &= \{(x,y) \in U \times V | x \notin A \text{ or } y \notin B\} \\ &= \{(x,y) | (x \in U \setminus A \text{ and } y \in V) \text{ or } (x \in U \text{ and } y \in V \setminus B)\} \\ &= ((U \setminus A) \times V) \cup (U \times (V \setminus B))\end{aligned}$$

Now

$$\begin{aligned}\partial(A \times B) &= \overline{A \times B} \setminus (A \times B)^\circ \\ &= (\overline{A} \times \overline{B}) \setminus (\mathring{A} \times \mathring{B}) \\ &= ((\overline{A} \setminus \mathring{A}) \times \overline{B}) \cup (\overline{A} \times (\overline{B} \setminus \mathring{B})) \\ &= ((\partial A) \times \overline{B}) \cup (\overline{A} \times (\partial B))\end{aligned}$$

■

11.4(c,d): Prove Proposition 11.7 (c,d)

(c): If $f : X \to Y$ is an injective continuous map of topological spaces and Y is Hausdorff, then so is X.
Solution: $x \neq y \in X$, then $f(x) \neq f(y) \in Y$. Since Y is Hausdorff there exist disjoint open sets $U \ni f(x)$ and $V \ni f(y)$. Then $f^{-1}(U)$ is a open set containing x and $f^{-1}(V)$ is an open set containing y (since f is continuous), and these sets are disjoint $(f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset))$. ■

(d): If X and Y are homeomorphic then X is Hausdorff iff Y is Hausdorrf. In other words, Hausdorffness is a topological property.
Solution: Let $f : X \to Y$ be a homeomorphism (bijective and f and f^{-1} are continuous. Apply part (c) as follows: if Y is Hausdorff use f. If X is Hausdorff use f^{-1}. ■

11.5: Suppose that $f : X \to Y$ is a continuous map of a topological space X to a Hausdorff space Y. Prove that the graph G_f of f is a closed subset of the topological product $X \times Y$.
Solution: We will show that $(X \times Y) \setminus G_f$ is open. Let $(a, b) \in (X \times Y) \setminus G_f$. Then $f(a) \neq b$, thus, since Y is Hausdorff, there exists disjoint open sets $U, V \in \mathcal{T}_Y$ such that $b \in U$ and $f(a) \in V$. Let $W = f^{-1}(\overline{U})$. This is a closed set in X since f is continuous. Therefore $X \setminus W$ is open in X, and we have $(X \setminus W) \times U$ is open in $X \times Y$, and it is contained in $(X \times Y) \setminus G_f$. This shows that $(X \times Y) \setminus G_f$ is open. ■

11.6: (a) Prove that if x is any point in a Hausdorff space X, then the intersection of all open subsets of X containing x is $\{x\}$.

Solution: Assume that there is a distinct element y in the intersection. Since X is Hausdorff there exist disjoint open sets $U \ni x$ and $V \ni y$. This is a contradiction. ■

(b) Give an example to show that the conclusion of (a) does not imply that X is Hausdorff. [**Hint: Think about the co-finite topology on an infinite set.**]

Solution: Let $X = \mathbb{Z}$ have the cofinite topology and let V be the intersection of all open sets containing a point $x \in X$. Now for each $y \neq x$ in X, $U_y = X \setminus \{y\}$ is an open set containing x. Now $\{x\} \subseteq V \subseteq \bigcap_{y \neq x} U_y = X \setminus \bigcup_{y \neq x} \{y\} = \{x\}$. But X is not Hausdorff (Example 11.6). ■

11.8: Suppose that X, Y are spaces, with Y Hausdorff, and that A is a subspace of X. Prove that if $f, g : \overline{A} \to Y$ are continuous and $f(a) = g(a)$ for all $a \in A$ then $f = g$.

Solution: Assume that there is an $x \in \overline{A} \setminus A$ such that $f(x) \neq g(x)$. Since Y is Hausdorff there exists disjoint open sets $U \ni f(x)$ and $V \ni g(X)$. Now $f^{-1}(U) \cap g^{-1}(V)$ is an open set since f and g are continuous, and it contains x which is in $\overline{A} \setminus A$ so $(f^{-1}(U) \cap g^{-1}(V)) \cap A \neq \emptyset$. Thus there is a $y \neq x$ in this set and $f(y) \in U$ and $g(y) \in V$. But since $y \in A$ we have $f(y) = g(y)$ which contradicts U and V being disjoint. ■