Bishop's University

Math 431: Test
Winter 2017

- Prepare neat solutions. Briefly justify your work, that is, *make your reasoning clear.*
- This test is 80 minutes in length. There are 37 points in total.

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1. (5 points) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x, 1-x)$. Describe the set $f^{-1}(D)$ where $D = \{(x,y)|x^2 - y \leq 1\}$.

> **Solution:** We require that $x^2 - y \leq 1$, so $x^2 - (1-x) = x^2 + x - 1 \leq 1$ which means we need to solve the inequality $x^2 + x - 2 = (x-1)(x+2) \leq 0$. Since the left side represents a upward opening parabola, the inequality is satified for the x values between the two roots. Thus $f^{-1}(D) = [-2, 1]$.

2. (5 points) Show that $d(x,y) = \left|x^3 - y^3\right|$ defines a metric on $\mathbb{R}$.

> **Solution:**
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> M1: $d(x,y) \geq 0$ by definition, and if $d(x,y) = 0$ then $abs x^3 - y^3 = 0$. The only way for this to happen is for $x^3 = y^3$, and thus $x = y$.
>
> M2: $d(x,y) = \left|x^3 - y^3\right| = \left|-(y^3 - x^3)\right| = \left|y^3 - x^3\right| = d(y,x)$.
>
> M3; $d(x,y) = \left|x^3 - y^3\right| = \left|x^3 - z^3 + z^3 - y^3\right| \leq \left|x^3 - z^3\right| + \left|z^3 - y^3\right| = d(x,z) + d(z,y)$.

3. Suppose that $(A, \mathcal{T}_A)$ is a subspace of a space $(X, \mathcal{T})$, and let $W \subseteq A$.

   (a) (5 points) Prove that if W is closed in A and A is closed in X, then W is closed in X.

   > **Solution:** We will show that the closure of W in X is W. Suppose x is a point of closure of W in X. Since $W \subseteq A$, x is also a point of closure of A in X. Since A is closed in X, we must have that $x \in A$. Since W is closed in A, we must have that $x \in W$. Thus $\overline{W} \subseteq W$ and since $W \subseteq \overline{W}$ by definition, we have that W is closed in X.

   (b) (2 points) Give an example of a proper subset $W \subset A$ where W is closed in A but W is not closed in X.

   > **Solution:** $X = \mathbb{R}$, $A = (0, 2)$, $W = (0, 1]$.

4. (5 points) Suppose that A is a subset of a topological space X. Prove that the boundary ∂A is closed in X.

Solution: $\partial A = \overline{X \setminus A} \cap \overline{A}$, so $X \setminus \partial A = (X \setminus (\overline{X \setminus A})) \cup (X \setminus \overline{A})$. Both $X \setminus (\overline{X \setminus A})$ and $X \setminus \overline{A}$ are open by definition, so their union is open. Thus ∂A is closed in$X$.

5. (5 points) Show that the intersection of two topologies on the same set X is also a topology on X, but that their union may or may not be a topology.

Solution: Let $\mathcal{T}_1, \mathcal{T}_2$ be two topologies on X, and define $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2$.

T1: Since $\emptyset, X \in \mathcal{T}_1, \mathcal{T}_2$ they must be in the intersection, $\mathcal{T}$.

T2: Let $A, B \in \mathcal{T}$, then $A, B \in \mathcal{T}_1, \mathcal{T}_2$. Since these are topologies, we have $A \cap B \in \mathcal{T}_1, \mathcal{T}_2$, and is thus in $\mathcal{T}$.

T3: Let $A_i \in \mathcal{T}$ for $i \in I$. Then $A_i \in \mathcal{T}_1, \mathcal{T}_2$. Since these are topologies, we have $\bigcup_{i \in I} A_i \in \mathcal{T}_1, \mathcal{T}_2$, and is thus in $\mathcal{T}$.

To show that the union of topologies may not be a topology, consider the following: $X = \{0, 1, 2\}$, $\mathcal{T}_1 = \{\emptyset, \{0\}, X\}$ and $\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{0\}, \{1\}, X\}$. This is not a topology since $\{0\} \cup \{1\} = \{0, 1\}$ is not in the union.

6. (5 points) Prove that any subspace of a Hausdorff space is Hausdorff.

Solution: Let $(X, \mathcal{T})$ be Hausdorff and $A \subseteq X$. Let $x \neq Y \in A \subseteq X$. Then there exist $U_x, U_y \in \mathcal{T}$ with $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$ since X is Hausdorff. Now $A \cap U_x, A \cap U_y \in \mathcal{T}_A$ with $x \in A \cap U_x$ and $y \in A \cap U_y$. We just have to show the intersection is empty. $(a \cap U_x) \cap (A \cap U_y) = A \cap (U_x \cap U_y) = A \cap \emptyset = \emptyset$. Thus $(A, \mathcal{T}_A$ is Hausdorff.

7. (5 points) Prove or disprove: If $f : X \to Y$ is a continuous map of topological spaces and $A \subseteq X$ is closed, then $f(A) \subseteq Y$ is closed.

Solution: The statement is not true. For example: Let $X = \mathbb{R}^2$ and $A = \{(x, \frac{1}{x}) | x > 0\}$. A is closed in $\mathbb{R}^2$.[1] Let $f : X \to \mathbb{R}$ be the projection onto the first coordinate. Projection maps are continuous. Then $f(A) = (0, \infty)$ which is open in $\mathbb{R}$.